RIEMANNIAN ORBIFOLDS WITH NON-NEGATIVE CURVATURE

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ABSTRACT

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Recent years have seen an increase in the study of orbifolds in connection to Riemannian geometry. We connect this field to one of the fundamental questions in Riemannian geometry, namely, which spaces admit a metric of positive curvature? We give a partial classification of 4 dimensional orbifolds with positive curvature on which a circle acts by isometries. We further study the connection between orbifolds and biquotients - which in the past was one of the main techniques used to construct compact manifolds with positive curvature. In particular, we classify all orbifold biquotients of $SU(3)$. Among those, we show that a certain 5 dimensional orbifold admits a metric of almost positive curvature. Furthermore, we provide some new results on the orbifolds $SU(3)//T^2$ studied by Florit and Ziller.

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Chapter 1

Introduction

First introduced by Satake [Sat56] as V-manifolds, orbifolds have come to be used in many different parts of mathematics. However, they did not come to prominence until Thurston rediscovered them (see [Thu80]) in trying to understand his famous Geometrization Conjecture, later proven using a different approach by Perelman [Per03].

Theorem (Geometrization Conjecture [Thu82], [Per03])**.** *The interior of every compact 3-manifold has a canonical decomposition into pieces which have geometric structures.*

Recall that orbifolds are locally modeled on $\mathbb{R}^n/\Gamma$ where Γ is a finite subgroup of $GL(n,\mathbb{R})$. Furthermore, if one desires a Riemannian metric on the orbifold, one may assume that $\Gamma \subset O(n)$. Since their introduction, orbifolds have found use in algebraic geometry (see for example [Dol82]), differential geometry of manifolds (see

[GWZ08]), understanding of tilings, and even mathematical modeling of music.

One important question in Riemannian geometry is what spaces admit metrics of positive curvature. In particular, the results that distinguish between manifolds admitting non-negative curvature and those admitting positive curvature are the theorems of Bonnet-Myers and Synge in the compact case and Perelman's proof of the soul conjecture in the non-compact case. If we add assumptions on the size of the isometry group, then we have the result of Hsiang and Kleiner [HK89], that a positively curved 4-dimensional Riemannian manifold with an isometric S^1 action is homeomorphic to either $S^4, \mathbb{RP}^4$ or $\mathbb{CP}^2$ (in fact by results of Fintushel [Fin78] this is true up to diffeomorphism). In higher dimensions, the assumption of a larger isometry group can be used to prove similar recognition theorems, see [GS94], [Wil03], [FR05], [Ken13] and [Wil07]. More recently, some work has been done on this question in a more general setting, see the work of Harvey and Searle [HS12] and Galaz-Garcia and Guijarro [GGG13] for results on positively curved Alexandrov spaces. In this thesis we will focus on the results dealing with orbifolds admitting metrics of positive, almost positive, i.e. positive on an open dense set, quasi-positive, i.e. non-negative and positive somewhere, and non-negative sectional curvature. A major open questions in this direction for manifolds is the Hopf Conjecture.

Conjecture (Hopf Conjecture). *The manifold $S^2 \times S^2$ does not admit a metric of positive sectional curvature.*

The best result in this direction comes from the work of Hsiang and Kleiner

Theorem (Hsiang-Kleiner [HK89]). *Let M be a 4-dimensional manifold with positive sectional curvature on which S^1 acts by isometries, then M is homeomorphic to either $S^4, \mathbb{CP}^2$ or $\mathbb{RP}^4$.*

In particular, this theorem implies that if the Hopf conjecture is false, then the counterexample must have a finite isometry group. We provide an orbifold analog to this result

Theorem A. *If $\mathcal{O}$ is a 4-dimensional S^1-orbifold with positive sectional curvature and $\pi_1^{orb}(\mathcal{O}) = 0$, $|\mathcal{O}|$ has $\mathbb{R}$-valued cohomology of either S^4 or $\mathbb{CP}^2$, and if there is a 2-dimensional component of the fixed point set, then we have an S^1-equivariant homeomorphism $|\mathcal{O}| = S^4$ or $|\mathcal{O}| = |\mathbb{CP}^2[\lambda]|$, otherwise, the S^1 action must have either 2 or 3 isolated fixed points.*

We also examine one of the techniques for constructing new examples of spaces of positive curvature - biquotients. Our work on this topic is partially motivated by the work of DeVito [DeV11] on classifying low-dimensional biquotients. While there are only finitely many diffeormorphism classes of biquotients in low dimension, there are infinitely many orbifold biquotients, even in dimension 2. Therefore, we will focus on orbifold biquotients of a single group, namely $SU(3)$. The choice of $SU(3)$ is motivated partially by historical reasons, since the first family of biquotients with positive curvature was constructed as $SU(3)//S^1$ by Eschenburg [Esc84], [Esc82], but also by the fact that $SU(3)$ is the lowest-dimensional Lie group whose orbifold biquotients have not been systematically studied, $SU(2)$ is low dimensional, and its biquotients are all of the form $\mathbb{CP}^1[\lambda_0, \lambda_1]$, and $S^3 \times S^3$ was studied by Kerin [Ker08].

Theorem B. *The following is a complete list of orbifold biquotients of the form $SU(3)//U$ with U connected:*

1. *Homogeneous spaces:*

   Classical manifolds $S^5 = SU(3)/SU(2)$, and $\mathbb{CP}^2 = SU(3)/U(2)$

   Wallach spaces $W^7_{p,q} = SU(3)/S^1_{p,q}$, $W^6 = SU(3)/T^2$

   The Wu manifold $SU(3)/SO(3)$

2. *Generalized Eschenburg spaces and orbifolds, of the form $SU(3)//S^1$ and $SU(3)//T^2$*

3. *Weighted projective spaces $SU(3)//(SU(2) \times S^1)$ and $SU(3)//U(2)$*

4. *Circle quotients of the Wu manifold $S^1_{p,q}\backslash SU(3)/SO(3)$*

5. *One orbifold of the form $SU(3)//SU(2)$.*

Of particular interest are the new family $S^1_{p,q}\backslash SU(3)/SO(3)$ and the exceptional example $SU(3)//SU(2)$. In section 5.2 we will study the orbifold structure of all of these, and the curvature properties of $SU(3)//SU(2)$. We will show that

Theorem C. *The orbifold $\mathcal{O}^5 = SU(3)//SU(2)$ admits a metric with almost positive curvature such that*

1. *The set of points with 0-curvature planes forms a totally geodesic, flat 2-torus T which is disjoint from the singular locus.*

2. *The only 0-curvature planes are those tangent to T.*

As a corollary we get a new example of an Alexandrov space with positive curvature:

Corollary 1.0.1. *The Alexandrov space $X^4 = \mathcal{O}^5/S^1 = SU(3)//U(2)$ admits a metric of positive sectional curvature.*

Also, we study the generalized Eschenburg spaces. In particular, we will focus on the orbifold structure of both 6 and 7 dimensional families; for the 6-dimensional family, we will provide corrections and improvements to the work of Florit and Ziller [FZ07]. The following theorem is the corrected and improved version of Theorem C of [FZ07].

Recall that E_d^7 is the family of 7-dimensional cohomogeneity one Eschenburg spaces $E^7_{(1,1,d),(0,0,d+2)}$, that is $SU(3)//S^1$ given by

$$z \star A = \mathrm{diag}(z, z, z^d) \cdot A \cdot \mathrm{diag}(1, 1, \overline{z}^{d+2}),$$

see [FZ07], [GWZ08] and [Zil09].

Theorem D. *Let E_d be a cohomogeneity one Eschenburg manifold, $d \geq 3$, equipped with a positively curved Eschenburg metric. Then:*

i) If S^1 acts on E_d^7 by isometries, then there are at minimum 3 singular points, in particular, if exactly two C_σ's are singular, then the $\mathcal{L}_{ij}$ connecting them is also singular.

In the following particular examples the singular locus of the isometric circle action $S^1_{a,b}$ on E_d consists of:

ii) A smooth totally geodesic 2-sphere with orbifold group $\mathbb{Z}_{d+1}$ if $a = (0, -1, 1)$ and $b = (0, 0, 0)$;

iii) When $a = (0, 1, 1)$ and $b = (2, 0, 0)$, the singular locus consists of four point with orbifold groups $\mathbb{Z}_3, \mathbb{Z}_{d+1}, \mathbb{Z}_{d+1}, \mathbb{Z}_{2d+1}$, and the following orbifold groups on spheres:

If $3|(d+1)$, then the first 2 points are connected by a totally geodesic 2-sphere with orbifold group $\mathbb{Z}_3$.

If $3|(d-1)$, then the first and the fourth points are connected by a totally geodesic 2-sphere with orbifold group $\mathbb{Z}_3$.

If $2|(d+1)$, then the second and the third points are connected by a totally geodesic 2-sphere with orbifold group $\mathbb{Z}_2$.

iv) A smooth totally geodesic 2-sphere with orbifold group $\mathbb{Z}_{d-1}$ if $a = (0, 1, 1)$ and $b = (0, 0, 2)$.

v) Three isolated singular points with orbifold groups $\mathbb{Z}_{2d-3}$, $\mathbb{Z}_{d^2-d-1}$, and $\mathbb{Z}_{d^2-d-1}$ if $a = (0, d-1, 0)$ and $b = (1, d-1, -1)$.

Furthermore, we prove

Theorem E. *Given an orbifold $\mathcal{O}^{a,b}_{p,q}$ which has positive curvature induced by a Cheeger deformation along $U(2)$, there exists $E^7_{u,v}$ (either a manifold or an orbifold) such that $\mathcal{O}^{a,b}_{p,q} = E^7_{u,v}//S^1$ and $E^7_{u,v}$ has positive curvature induced by Cheeger deformation along the same $U(2)$.*

Equivalently, there exist $\lambda, \mu \in \mathbb{Z}$ relatively prime such that $E^7_{\lambda p+\mu a, \lambda q+\mu b}$ is positively curved.

The exposition is organized as follows:

In Chapter 2, we start with a formal definition on an orbifold. We will also provide a number of well-known examples, and study a new family of examples. We will finish the chapter with an overview of basic results needed for the study of orbifolds, including new proofs of theorems of Bonnet-Myers and Synge for orbifolds. The main tool used in this section comes from a powerful, but simple result in Proposition 2.3.2 on the behavior of length minimizing geodesics on orbifolds.

In Chapter 3, we will provide background on biquotients, a powerful tool for constructing new examples of manifolds and orbifolds with non-negative curvature. We further provide an overview of a classical approach to improving the curvature properties by using Cheeger deformations. We also discuss some question in the theory of positive curvature, almost positive curvature and quasi-positive curvature. Recall that a metric is said to have almost positive curvature if the set of points with all sectional curvatures positive is dense, and quasi-positive if the sectional curvature is non-negative and there exists at least one point with all sectional curvatures strictly positive. An example by Wilking [Wil02] on $\mathbb{RP}^2 \times \mathbb{RP}^3$ shows that a metric with almost positive curvature can in general not be deformed to positive curvature.

In Chapter 4, we prove Theorem A. In proving this theorem, we examine some general properties of 4-dimensional orbifolds, and specifically those which admit an orbifold-smooth S^1 action.

In Chapter 5, we prove Theorem B, and examine the structure of the new examples, leading to a proof of Theorem C and Corollary 1.0.1. We also examine the structure of biquotients $SU(3)//T^2$ leading to the proof of Theorem D. Additionally, we also prove Theorem E in this chapter.

Chapter 2

Orbifolds

2.1 Fundamentals of Orbifolds

We begin this section by discussing the formal definition of an orbifold.

Definition 2.1.1. An n-dimensional orbifold $\mathcal{O}$ is a paracompact Hausdorff space together with an open cover by sets U_i that form an atlas as defined below.

An orbifold chart is a triple $(U_i, \Gamma_i, \varphi_i)$, with Γ_i a finite subgroup of $O(n)$ (in general $\Gamma_i \subset GL(n, \mathbb{R})$ is also acceptable, but with a proper choice of a metric, the distinction is irrelevant), and a homeomorphism $\varphi : \widetilde{U_i}/\Gamma_i \xrightarrow{\sim} U_i$ with $\widetilde{U_i} \subset \mathbb{R}^n$.

An orbifold atlas $\{(U_i, \Gamma_i, \varphi_i)\}$ is a collection of charts satisfying

1. If $U_i \cap U_j \neq 0$, then there exists a chart $(U_k, \Gamma_k, \varphi_k)$ such that $U_k = U_i \cap U_j$.

2. Furthermore, the following diagram commutes, where the maps $\varphi_* \widetilde{U_*}/\Gamma_* \to U_*$ are homeomorphisms, and the maps $f_{k*} : U_k \to U_*$, $\widetilde{f_{k*}} : \widetilde{U_k} \to \widetilde{U_*}$ are

injective continuous and smooth maps respectively.

$$
\begin{array}{ccccc}
\widetilde{U_i} & \hookleftarrow & \widetilde{U_k} & \hookrightarrow & \widetilde{U_j} \\
\downarrow & & \downarrow & & \downarrow \\
\widetilde{U_i}/\Gamma_i & \hookleftarrow & \widetilde{U_k}/\Gamma_k & \hookrightarrow & \widetilde{U_j}/\Gamma_j \\
\downarrow \cong & & \downarrow \cong & & \downarrow \cong \\
U_i & \hookleftarrow & U_k & \hookrightarrow & U_j
\end{array}
$$

Remark 2.1.1. There are some immediate consequences of this definition, namely:

1. Whenever $U_k \subset U_i$, there exists an injective group homomorphism $\psi_{ki} : \Gamma_k \to \Gamma_i$.

2. If $p \in U_i, U_j$, then given a lifts of p to $\widetilde{U_i}$ and $\widetilde{U_j}$, then stabilizers of thee lifts under the actions of Γ_i, Γ_j respectively are conjugate subgroups of $O(n)$.

We call the sets $\widetilde{U_i}$ the local covers of U_i. Furthermore, if we require that Γ_i preserve the metric on $\widetilde{U_i}$, and assume that the metric glues together, that is the embeddings $f_{ki} : \widetilde{U_k} \hookrightarrow \widetilde{U_i}$ are isometric immersions. As a result, we can define a geodesic on $\mathcal{O}$ to be locally the image of a geodesic in the local manifold cover.

The above construction gives us a notion of an orbifold group at a point.

Definition 2.1.2. Let $x \in \mathcal{O}$, suppose $x \in U_i$. Then, the orbifold group at x, which we denote as Γ_x is defined to be the subgroup of Γ_i which fixes a given lift of x to $\widetilde{U_i}$.

Note that this definition is well behaved due to the fact that the stabilizer of a point is independent from the choice of chart containing that point. Furthermore, the group is only defined up to conjugation, which corresponds to taking different lifts.

Ocasionally, we will want to ignore the orbifold structure of $\mathcal{O}$, in this case, we will consider the underlying topological space of $\mathcal{O}$, which we denote $|\mathcal{O}|$. This is the paracompact Hausdorff space used in the definition of an orbifold, with no orbifold charts. On the other hand, we will ocasionally want to decompose $\mathcal{O}$ into pieces with the same orbifold group. In this case, we will view $\mathcal{O}$ as a disjoint union of strata $\mathcal{S}$ where each $\mathcal{S}$ is a connected component with a constant (up to conjugation) orbifold group. Each stratum is a topological manifold. One stratum deserves special mention, and that is $\mathcal{O}^{reg}$ which is the stratum consisting of regular points, that is points where $\Gamma_x = \{e\}$. This stratum is a connected manifold which is dense in $|\mathcal{O}|$; all other strata have positive codimension.

The strata decomposition gives us a simple definition of the orbifold Euler characteristic:

Definition 2.1.3. Let $\mathcal{A}$ be the set of strata of an orbifold $\mathcal{O}$. Then, the orbifold Euler characteristic can be defined as

$$\chi^{orb}(\mathcal{O}) = \sum_{\mathcal{S}\in\mathcal{A}} \frac{\chi(\mathcal{S})}{|\Gamma_{\mathcal{S}}|},$$

where $\Gamma_{\mathcal{S}}$ is the orbifold group along the stratum $\mathcal{S}$.

Note that in the degenerate case where $\mathcal{O}$ is itself a manifold, $\chi^{orb}(\mathcal{O}) = \chi(\mathcal{O})$.

Let $\mathcal{O}$,$\mathcal{U}$ be two orbifolds, with atlases U_i, V_j respectively, we say that a map $\Psi : \mathcal{O} \to \mathcal{U}$ is a smooth orbifold map if there exist refinements of the atlases, which we call U_i', V_j' such that Ψ induces maps $\psi_{ij} : U_i' \to V_j'$ which lifts to a smooth map $\widetilde{U_i'} \to \widetilde{V_j'}$ (structurally, V_j is a quotient of U_i). We say that Ψ is an orbifold diffeomorphism if there exists a smooth orbifold map $\Psi^{-1} : \mathcal{U} \to \mathcal{O}$, with $\Psi^{-1} \circ \Psi = id : \mathcal{O} \to \mathcal{O}$ and $\Psi \circ \Psi^{-1} = id : \mathcal{U} \to \mathcal{U}$. Since we get V_j is a quotient of U_i and U_i a quotient of V_j, an orbifold diffeomorphism must preserve strata. Similarly, an orbifold covering map is a smooth orbifold map that is also a branched cover.

We also recall the definition of π_1^{orb}, the orbifold fundamental group. There are two equivalent ways of defining it. The first using universal covers, that is if $\mathcal{O} = \mathcal{U}/\Gamma$, Γ discrete and $\mathcal{U}$ admitting no covers, then $\pi_1^{orb}(\mathcal{O}) = \Gamma$. The second method is via homotopy, that is, given two loops $\gamma_1, \gamma_2 : S^1 \to \mathcal{O}^{reg}$, we say they are orbifold homotopic if γ_1 can be homotoped to γ_2 via regular homotopies inside $\mathcal{O}^{reg}$ and end-point preserving homotopies in the local manifold covers of neighborhoods of singular points. Then, we define $\pi_1^{orb}(\mathcal{O})$ to be the group of loops in $\mathcal{O}^{reg}$ modulo base point preserving orbifold homotopy. For computation, the way to view this is $\pi_1(\mathcal{O}^{reg})/N$, where N is the normal subgroup generated by the images of contractible loops in the local covers $\widetilde{\mathcal{U}_i}$. For explicit construction in the two-dimensional case, see [Sco83].

Additionally, we say that an orbifold $\mathcal{O}^n$ is orientable if $\mathcal{O}^{reg}$ is an orientable manifold, and for each $x \in \mathcal{O}$, $\Gamma_x \subset SO(n)$. If an orbifold $\mathcal{O}$ is orientable, choosing an orientation of $\mathcal{O}$ corresponds to choosing an orientation on $\mathcal{O}^{reg}$. Finally, we call an orbifold good if it is a quotient of a manifold by a discrete group, and we call it bad otherwise.

2.2 Examples

We start this section with the simplest examples, namely the 2-dimensional orbifolds. For two dimensional orbifolds, there are three types of singularities that can occur:

- A cone singularity, when the orbifold group is a cyclic group of rotations.
- A mirror singularity, when the orbifold group is a reflection about some line. This type of singularity looks like boundary on the level of the underlying space.
- A corner singularity, when the orbifold group is a dihedral group consisting of rotations and reflections.

In particular, we examine the classification of bad 2-dimensional orbifolds

Proposition 2.2.1 ([Sco83]). *Let $\mathcal{O}^2$ be a complete 2-dimensional bad orbifold, then $\mathcal{O}$ is one of the following:*

1. *A "teardrop" - an orbifold with S^2 as its underlying space and one cone point singularity, with orbifold group $\mathbb{Z}_k$ ($k > 1$).*
2. *A "lemon" or "football" - an orbifold with S^2 as its underlying space and two cone point singularities, with orbifold groups $\mathbb{Z}_p, \mathbb{Z}_q$ with $p \neq q$ and $p, q > 1$.*

3. *A disc with one corner - an orbifold with D^2 as its underlying space, a mirror singularity along the boundary, and one corner singularity, with orbifold group D_k ($k > 1$).*

4. *A disc with two corners - an orbifold with D^2 as its underlying space, a mirror singularity along the boundary, and two corner singularities, with orbifold groups D_p, D_q with $p \neq q$ and $p, q > 1$.*

The main idea of this classification is to show that "teardrop" orbifolds and "lemon" orbifolds with $(p, q) = 1$ are the only 2-orbifolds with $\pi_1^{orb} = 0$, that are not manifolds. The remaining cases are obtained as quotients of these two.

Example 2.2.1. An example of a good 2-orbifold is $\mathcal{O} = S^2(2, 3, 4) = S^2/S_4$. This example is obtained by taking the quotient of S^2 by the symmetry group of a cube (or octahedron). The resulting orbifold has S^2 as its underlying space, and 3 singular points of order 2, 3 and 4. These correspond to the centers of edges, vertices and centers of faces of the cube respectively.

Similarly, by considering other finite subgroups of $SO(3)$, one can obtain the orbifolds $S^2(n, n) = S^2/\mathbb{Z}_n$, $S^2(2, 2, n) = S^2/D_n$, $S^2(2, 3, 3) = S^2/A_4$, and $S^2(2, 3, 5) = S^2/A_5$. Where $\mathbb{Z}_n$ is the symmetry group of a regular n-sided pyramid, D_n is the symmetry group of a regular n-sided prism, A_4 is the symmetry group of a tetrahedron, and A_5 is the symmetry group of a dodecahedron (or icosahedron).

Example 2.2.2. A very useful and common family of examples are known as the weighted projective spaces. These are bad orbifolds given by quotients of the form

S^{2n+1}/S^1 similar to the standard complex projective spaces. The properties of this family of spaces make it of interest to both differential and algebraic geometers, see for example [Dol82].

To obtain a weighted projective space $\mathbb{CP}^n[\lambda_0, \lambda_1, \ldots, \lambda_n]$, where λ_i are all positive integers with $\gcd(\lambda_0, \lambda_1, \ldots, \lambda_n) = 1$, one modifies the standard action of S^1 on $\mathbb{C}^{2n+2}$. Namely, $z \star (w_0, w_1, \ldots, w_n) = (z^{\lambda_0} w_0, z^{\lambda_1} w_1, \ldots, z^{\lambda_n} w_n)$. In particular, if $\lambda_0 = \lambda_1 = \cdots = \lambda_n = 1$, then one obtains the standard $\mathbb{CP}^n$.

For brevity, we will often denote a weighted projective space as $\mathbb{CP}^n[\lambda]$ when the exact values of λ_i's are not necessary.

In Chapters 4 and 5 we will focus mostly on $\mathbb{CP}^2[\lambda_0, \lambda_1, \lambda_2]$, so in this section, we will focus on the general properties.

The singular locus of $\mathbb{CP}^n[\lambda]$ consists of up to $\binom{n}{k}$ copies of $\mathbb{CP}^k[\lambda']$ for $0 \leq k < n$. These correspond to the coordinate $(k+1)$-dimensional subspaces of $\mathbb{C}^{n+1}$. In particular, the suborbifold $\mathbb{CP}^k[\lambda']$ corresponding to the subspace spanned by $z_{i_0}, \ldots, z_{i_k}$ is singular iff $l = \gcd(\lambda_{i_0}, \ldots, \lambda_{i_k}) > 1$, and then the general orbifold group along this suborbifold is cyclic of order l.

For example, $\mathbb{CP}^1[p, q]$ is the 2-dimensional p, q "lemon"-shaped orbifold if $p, q > 1$ and the p or q "teardrop" if $q = 1$ or $p = 1$ respectively.

2.2.1 Weighted $\mathbb{HP}^n$

After seeing the weighted projective spaces above, one might inquire whether a similar construction can be achieved for quaternionic projective spaces. If one interprets the choice of λ_i's above as a choice of a representation of S^1, then one must ask what representations of $S^3 = SU(2) = Sp(1)$ have nice behavior on a sphere. The answer comes from the observation that any odd-dimensional complex representation of $SU(2)$ has isotropy groups of rank 1. Namely, the only representations that induce foliations of S^k by leaves of constant dimension come from direct sums of the irreducible quaternionic represesntations of $Sp(1)$. Alternatively, one can think of these as direct products of irreducible even-dimensional complex representations of $SU(2)$.

Definition 2.2.1. A weighted quaternionic projective space $\mathbb{HP}^n[d_1, \ldots, d_k]$, with $d_i > 0$ and $\sum d_i = n+1$, is the quotient of $S^{4n+3} \subset \mathbb{H}^{n+1} = \mathbb{H}^{d_1} \oplus \cdots \oplus \mathbb{H}^{d_k}$ by the left-action of $S^3 = Sp(1)$ given by $\rho = \rho_{d_1} \oplus \cdots \oplus \rho_{d_k}$, where ρ_d is the irreducible d-dimensional quaternionic representation ($4d$-dimensional real representation) of $Sp(1)$.

The simplest example of a non-trivial weighted quaternionic projective space is $\mathbb{HP}^1[2]$. One way to view this orbifold is as $Sp(1)\backslash Sp(2)/Sp(1)^{pr}$, where the group on the right corresponds to the irreducible 2-dimensional quaternionic representation, and the group on the left corresponds to the embedding $\{\text{diag}(q, 1) | q \in Sp(1)\} \subset Sp(2)$. In particular, the quotient $Sp(2)/Sp(1)^{pr}$ is diffeomorphic to

$B^7 = SO(5)/SO(3)$ (the 7-dimensional Berger space [Ber61]), and the $Sp(1)$ acting on the left corresponds to an $S^3 \subset SO(4)$, where the action of $SO(4)$ gives a cohomogeneity one structure on B^7 (see [GWZ08]). Looking at the isotropy groups of the $SO(4)$ action, we can observe that $\mathbb{HP}^1[2]$ is a cohomogeneity one orbifold with an $SO(3)$ action, and the underlying space is S^4 with a $\mathbb{Z}_3$ singularity along a Veronese $\mathbb{RP}^2$. In particular, as we will see in Chapter 4, this is the Hitchin orbifold $\mathcal{H}_3$.

One interesting property of the orbifold $\mathbb{HP}^1[2]$ is that it admits positive curvature since it is the base of a Riemannian submersion of a positively curved total space. Furthermore, Hitchin [Hit96] showed that it admits a self-dual Einstein metric. Finally, two recent papers ([GVZ11] and [Dea11]) have shown that there is a 7-dimensional manifold P_2 obtained as $S^3 \to P_2 \to \mathbb{HP}^1[2]$ (see [GWZ08] for the construction), which admits a metric of positive curvature compatible with this fibration. In particular, there are 3 different orbi-fiber bundles with fiber S^3, and base space $\mathbb{HP}^1[2]$, where the total space is a positively curved manifold (S^7, P_2, B^7).

One might inquire whether there are other 7-dimensional manifolds with positive curvature which fiber over this orbifold. Unfortunately, the study of bundles over orbifolds is not well-developed at present.

2.3 Basic Results

In this section we prove several known orbifold results. In particular, using Proposition 2.3.2, we are able to greatly simplify the work of Borzellino on the orbifold Bonnet-Myers theorem.

The following is a simple yet very useful proposition for dealing with orbifolds. A more general version is proven by Armstrong in [Arm68].

Proposition 2.3.1. *Let X be a simply-connected topological space. Let Γ be a finite group acting continuously on X. Then, $\pi_1(X/\Gamma) = \Gamma/\Gamma_f$ where Γ_f is generated by the elements of Γ that have fixed points.*

For our purposes, the proposition above will use $X = S^{n-1}$ with $n > 2$, since it in particular allows us to compute $\pi_1(\partial(B^n/\Gamma_x)) = \pi_1(S^{n-1}/\Gamma_x)$ the fundamental group of the boundary of a neighborhood of a singular point. In particular, when $n = 4$, and $\Gamma_x \subset SO(4)$ we get S^{n-1}/Γ_x is a topological manifold (this comes from the fact that every orientable orbifold quotient of S^2 is homeomorphic to S^2, and therefore, if $\pi_1(S^3/\Gamma_x) = 0$, then it must be S^3 by the Poincaré Conjecture.

The following is a generalization of a well-known Lemma due to Kleiner [Kle90] in the manifold case. It can be proven as a corollary of Kleiner's work, since an orbifold is the quotient of the orthonormal frame bundle, which is a smooth manifold, by the action of $SO(n)$. However, we provide a direct proof.

Proposition 2.3.2. *Let $\mathcal{O}^n$ be a complete Riemannian orbifold, $p, q \in \mathcal{O}$, γ :*

$[0,t] \to \mathcal{O}$ *a minimizing geodesic with* $\gamma(0) = p, \gamma(t) = q$*. Let* $k = \text{codim } \mathcal{S}(p)$*,* $l = \text{codim } \mathcal{S}(q)$*, then* γ *maps* $(0,t)$ *to a single stratum,* $\mathcal{S}_0$*, where* $\text{codim } \mathcal{S}_0 \leq \min\{k,l\}$.

Proof. Suppose $\gamma(s)$ lies in a higher codimension startum than $\gamma(s-\varepsilon)$ for small $\varepsilon > 0$. Let $m = \text{codim } \mathcal{S}(\gamma(s))$.

Take a neighborhood of $\gamma(s)$ that has the form $N = B^{n-m}(\delta) \times B^m(\eta)/\Gamma$, where $\Gamma \subset O(m) \subset O(n)$ is the orbifold group along $\mathcal{S}(\gamma(s))$. We can choose $\delta, \eta > 0$ small such that $\gamma(s-\tau) \in B^{n-m}(\delta) \times S^{m-1}(\eta)/\Gamma$ is the point where γ enters N.

Consider a lift $\widetilde{\gamma} : (s-\tau, s+\lambda) \to \widetilde{N}$ where $\widetilde{N}$ is the local manifold cover of N. Since γ is length minimizing, as such, we can take $\widetilde{\gamma}$ to be length minimizing as well.

Define $\widehat{\gamma}_g : (-\tau, \lambda) \to \widetilde{N}$ where $g \in \Gamma, g \neq I$ fixes $\widetilde{\gamma}(s)$ as follows:

$$\widehat{\gamma}_g(x) = \begin{cases} g \circ \widetilde{\gamma}(s+x) & x < 0, \\ \widetilde{\gamma}(s+x) & x \geq 0 \end{cases}$$

$\widehat{\gamma}_g$ is continuous since $g \circ \widetilde{\gamma}(s) = \widetilde{\gamma}(s)$, it has the same image in N as $\widetilde{\gamma}$; however, it is not a geodesic of $\widetilde{N}$. Therefore, we can shorten it while keeping the same endpoints. This implies that γ can be shortened inside N. Which contradicts our assumption that γ is a minimizing geodesic. □

Corollary 2.3.3. *If* $p, q \in \mathcal{O}^{\text{reg}}$*, then the shortest distance between* p *and* q *can only be achieved by a geodesic segment contained entirely in* $\mathcal{O}^{\text{reg}}$*.*

We can use this corollary to obtain a simple proof of the following result (see [Bor93] where this was obtained via volume comparison):

Corollary 2.3.4 (Bonnet-Myers for Orbifolds)**.** *Let $\mathcal{O}^n$ be a complete Riemannian orbifold. Suppose that the Ricci curvature of $\mathcal{O}$ satisfies $Ric_p(v) \geq \frac{1}{r^2} > 0$ for all $p \in \mathcal{O}$ and for all $v \in T_p\mathcal{O}$. Then $\mathcal{O}$ is compact and the diameter* $\operatorname{diam} \mathcal{O} \leq \pi r$*.*

Proof. Suppose $\operatorname{diam} \mathcal{O} > \pi r$. Since the set of regular points is dense, there exist $p, q \in \mathcal{O}^{\mathrm{reg}}$ such that $d(p, q) > \pi r$. Let γ be a length minimizing geodesic segment connecting p and q. By the previous corollary, we know that γ lies in $\mathcal{O}^{\mathrm{reg}}$. Moreover, since $\mathcal{O}^{reg}$ is open, a neighborhood of γ lies in $\mathcal{O}^{\mathrm{reg}}$, which means we can use the standard proof of Bennet-Myers using the second variation formula. □

Theorem 2.3.5 (Synge-Weinstein for Orbifolds)**.** *Let f be an isometry of a compact oriented Riemannian orbifold $\mathcal{O}^n$. Suppose that $\mathcal{O}$ has positive sectional curvature and that if n is even f preserves orientation and if n is odd, f reverses orientation. Then, f has a fixed point.*

Proof. Let $d_f : \mathcal{O} \to \mathbb{R}$ be defined by $d_f(p) = d(p, f(p))$.

Let $p \in \mathcal{O}$ be a point that minimizes d_f. We assume that $d_f(p) > 0$.

Let γ be a minimizing geodesic connecting p to $f(p)$. By the same argument as the standard Synge-Weinstein theorem, f acts on γ by translation by $d_f(p)$. We assume that $\gamma(0) = p, \gamma(T) = f(p)$.

We now show that $\gamma : (-\infty, \infty)$ must lie in a single stratum.

Let $q = \gamma(T/3)$ and $r = \gamma(2T/3)$. By Proposition 2.3.2 we know that since γ is length minimizing on $[0, T]$, q, r are in the same stratum. Furthermore, γ must be length minimizing on $[T/3, 4T/3]$. Therefore, r and $f(p)$ are in the same stratum. From this we conclude that the entire image of γ is in a single stratum.

Let N be a δ-tubular neighborhood of $\gamma([-\varepsilon, T + \varepsilon])$. Then, for $\delta, \varepsilon > 0$ small enough, we can take a local manifold cover of N, which we will call $\widetilde{N}$. By taking the unique lift of $\gamma([0, T])$ to $\widetilde{N}$, we can apply the same steps as the proof of the standard Synge-Weinstein theorem to conclude that there exists a point q close to p such that $d_f(q) < d_f(p)$. $\square$

An interesting question is how to adapt Synge's theorem to the orbifold case. The straight forward adaptation fails. For example $\mathcal{O}^2 = S^2/\mathbb{Z}^2$, where $\mathbb{Z}^2$ acts by rotation by 180° is orientable, but $\pi_1^{orb}(\mathcal{O}) = \mathbb{Z}_2$, and $\mathcal{O}^3 = S^3/\mathbb{Z}_2$ where $\mathbb{Z}_2$ acts by reflections in some 3-plane in $\mathbb{R}^4$ is not orientable in the orbifold sense.

The corollary below demonstrates the analog of Synge's Theorem for Orbifolds, a version for Alexandrov spaces can be found in [HS12], and with some additional assumptions in [Pet98]

Corollary 2.3.6 (Synge's Theorem for Orbifolds)**.** *Let $\mathcal{O}$ be a compact positively curved orbifold, then*

1. *if n is even, and $\mathcal{O}$ orientable, then $|\mathcal{O}|$ is simply connected.*

2. *if n is odd, and for every $p \in \mathcal{O}$, $\Gamma_p \subset SO(n)$, then $\mathcal{O}$ is orientable.*

Proof. For the even dimensional case, we suppose that $|\mathcal{O}|$ is not simply connected, let $\mathcal{U}$ be a finite cover of $\mathcal{O}$ such that $|\mathcal{U}|$ is a cover of $|\mathcal{O}|$. Let f be a deck transformation, then f is an isometry of $\mathcal{U}$ that preserves orientation. By Synge-Weinstein, f must have a fixed point; therefore, $|\mathcal{U}|$ can not be a cover of $|\mathcal{O}|$.

For the odd dimensional case, we suppose $\mathcal{O}$ is non-orientable, then there is a double cover $\mathcal{U} \to \mathcal{O}$ with $\mathcal{U}$ orientable and positively curved. By Synge-Weinstein, the orientation reversing isometry of $\mathcal{U}$ has a fixed point. Furthermore, it is clear that the action on the tangent space at that point is orientation reversing, so the orbifold group at the image of this fixed point will not lie in $SO(n)$. □

Finally, we add in this section a few warnings about scenarios in which the behavior of orbifolds differs significantly from that of manifolds.

If a finite group fixes two complementary subspaces of $T_p\mathcal{O}$, then it need not fix the entire tangent space (see Example 4.1.3, and more detailed description of what occurs in this scenario in 4).

Additionally, there is a challenge in defining when two vectors is $T_p\mathcal{O}$ are orthogonal. The first possible definition is to say that v, w are orthogonal if there are lifts $\widetilde{v}, \widetilde{w}$ which are orthogonal in the local cover. However, with this definition, it is possible to have v orthogonal to itself, for example $\mathcal{O} = S^2/rot_{\pi/2}$ then at the north pole, every vector is orthogonal to itself. The second possible definition is to say that v, w are orthogonal if all lifts $\widetilde{v}, \widetilde{w}$ are orthogonal in the local cover. Under this definition, it is possible that v has no non-zero vectors orthogonal to it (same

$\mathcal{O}$ as before); nevertheless, this is the definition we prefer.

We now show that the slice theorem holds for orbifolds and respects the orbifold structure. Note that it is known to hold for Alexandrov spaces [HS12], although the proof does not describe the orbifold structure. In fact, in the orbifold case, the proof is a straightforward generalization of that for manifolds.

Proposition 2.3.7 (Slice Theorem for Orbifolds)**.** *Let G be a connected Lie group, $\mathcal{O}^n$ a Riemannian orbifold, with an isometric G-action. Then, given any $p \in \mathcal{O}$, and sufficiently small $r > 0$, we have*

$$B_r(G(p)) \cong G \times_{G_p} Cone(\nu_p)$$

where $G_p = \{g \in G | g(p) = p\}$, $G(p)$ is the orbit of p, and $\nu_p = \{v \in T_p\mathcal{O} | v \perp G(p), |v| = 1\}$ is the space of directions orthogonal to the orbit.

Remark 2.3.1. In the course of the proof we will also show that it does not matter which definition of orthogonality we use.

Proof. We begin with the observation that $G(p) \subset \mathcal{S}(p)$, since $g(p)$ must lie in a stratum with the same orbifold group, and $G(p)$ is connected. In particular, this tells us that $G(p)$ lifts uniquely to the local manifold cover at p, as $\mathcal{S}(p)$ must lie in $\mathrm{Fix}(\Gamma_p)$. Furthermore, with respect to this lift, $\Gamma_p \subset O(k) \subset O(n)$, where k is the codimension of $\mathcal{S}(p)$. Let l be the codimension of $G(p)$ inside $\mathcal{S}(p)$, then $\nu_p = S^{k+l-1}/\Gamma_p$ is well-defined.

These observations allow us to approach the proof for the orbifold case in the same fashion as the manifold case. We define $\varphi : G \times Cone(\nu_p) \to \mathcal{O}$ as

$$\varphi(g, v) = g(\exp_p(v)),$$

this map has kernel G_p, and so induces a map $G \times_{G_p} Cone(\nu_p) \to \mathcal{O}$. To get an inverse map, for q close to $G(p)$, we take q_0 to be the point on $G(p)$ closest to q, and $g_0 \in G$ be such that $g_0(p) = q_0$, this choice is unique up to G_p. We then consider $v_0 \in \mathrm{Cone}(\nu_p) \subset T_p\mathcal{O}$ such that $\exp_p(v_0) = g_0^{-1}(q)$. This gives us our diffeomorphism. □

We end the section with a result which appeared in the original paper of Satake:

Proposition 2.3.8 (Orbifold Poincaré Duality [Sat56]). *Let $\mathcal{O}$ be a compact orientable n-dimensional orbifold without boundary, then $\mathcal{O}$ is a $\mathbb{R}$-homology manifold, and in particular, $H^k(|\mathcal{O}|; \mathbb{R}) = H_{n-k}(|\mathcal{O}|; \mathbb{R})$.*

A direct proof of this result is not difficult, and relies on understanding the relation between $H^*(B^n, S^{n-1})$ and $H^*(B^n/\Gamma, S^{n-1}/\Gamma)$ when Γ is a finite group acting effectively, and preserving orientation (see [Gro57] and [Mac62]). In particular, the only difference between the two cohomology rings is in the torsion, which is killed off when passing to real coefficients.

Chapter 3

Group Actions and Metrics

3.1 Biquotients

Recall that a biquotient X is of the form $X = G//U$ where G is a Lie group and $U \subset G \times G$ acts as $(u_l, u_r) \cdot g = u_l g u_r^{-1}$ where $g \in G$ and $(u_l, u_r) \in U$. If $U = K \times H$ with $K, H \subset G$, then we can instead write $X = K\backslash G/H$. In his Habilitation, Eschenburg [Esc84] showed that if $G//U$ is a manifold, then $\operatorname{rk} U \leq \operatorname{rk} G$. Since the argument is done on the Lie algebra level, we will see that it also holds when we allow $G//U$ to be an orbifold.

Lemma 3.1.1. *Let $\mathfrak{t}_{\mathfrak{u}} \subset \mathfrak{u}$ be a maximal abelian subalgebra. Then, $G//U$ is an orbifold if and only if for all non-zero $(X_1, X_2) \in \mathfrak{t}_{\mathfrak{u}} \subset \mathfrak{u} \subset \mathfrak{g} \oplus \mathfrak{g}$ and for all $g \in G$, $X_1 - \operatorname{Ad}(g) X_2 \neq 0$.*

If $\pi : G \to G//U$ is the projection, and $g \in G$, then the orbifold group $\Gamma_{\pi(g)} \subset U$

is given by

$$\Gamma_{\pi(g)} = \left\{(h,k) \in U \mid hgk^{-1}g^{-1} = e\right\}.$$

In particular, $G//U$ is a manifold iff $(h,k) \in U, g \in G$ with $h = gkg^{-1}$ implies $h = k = e$.

Proof. Let M be a manifold, Γ a Lie group, $\pi : M \to \mathcal{O} = M/\Gamma$ the projection map. Then, for any $x \in M$, $\Gamma_{\pi(x)} = \mathrm{Stab}(x)$. As a corollary we get the description of $\Gamma_{\pi(g)}$ as desired. Observe that $G//U$ is an orbifold iff the stabilizer of every $g \in G$ is finite. The Lie algebra of $\Gamma_{\pi(g)}$ is $X_1 - \mathrm{Ad}(g)X_2$ where $(X_1, X_2) \in \mathfrak{u}$. Since we can conjugate $(X_1, X_2) \in \mathfrak{u}$ into an element of $\mathfrak{t}_{\mathfrak{u}} \subset \mathfrak{u}$, and since the stabilizer groups occur in conjugacy classes, the first claim follows as well. □

Lemma 3.1.2. *If $G//U$ is an orbifold, then $\mathrm{rk}\,\mathfrak{u} \leq \mathrm{rk}\,\mathfrak{g}$.*

Proof. Suppose $\mathrm{rk}\,\mathfrak{u} > \mathrm{rk}\,\mathfrak{g}$, let $\mathfrak{t}_{\mathfrak{u}} \subset \mathfrak{u} \subset \mathfrak{g} \oplus \mathfrak{g}$ be a maximal torus. Let φ_1, φ_2 be projections of $\mathfrak{u}$ onto the first and second copy of $\mathfrak{g}$ respectively. Pick a maximal torus $\mathfrak{t}_{\mathfrak{g}} \subset \mathfrak{g}$ such that $\varphi_1(\mathfrak{t}_{\mathfrak{u}}) \subset \mathfrak{t}_{\mathfrak{g}}$.

Next, pick $g \in G$ such that $\mathrm{Ad}(g)\varphi_2(\mathfrak{t}_{\mathfrak{u}}) \subset \mathfrak{t}_{\mathfrak{g}}$. This induces a linear map $\Phi : \mathfrak{t}_{\mathfrak{u}} \to \mathfrak{t}_{\mathfrak{g}}$ given by

$$\Phi(X) = \varphi_1(X) - \mathrm{Ad}(g)\varphi_2(X).$$

In particular, if $\mathrm{rk}\,\mathfrak{u} > \mathrm{rk}\,\mathfrak{g}$, we conclude that $\ker \Phi \neq \{0\}$. This implies that there exists $X = (\varphi_1(X), \varphi_2(X)) \in \mathfrak{u}$ such that $\varphi_1(X) - \mathrm{Ad}(g)\varphi_2(X) = 0$, which by the previous lemma implies that $G//U$ is not an orbifold. □

Remark 3.1.1. A special situation occurs when one deals with biquotients of the form $SU(n)//U$. In particular, one does not usually require that $U \subset SU(n) \times SU(n)$, but rather that $U \subset U(n) \times U(n)$, such that if $(g, h) \in U$, then $\det(g) = \det(h)$, which ensures that $SU(n)$ is a subset of $U(n)$ that is preserved under the action of U.

3.1.1 Classical Examples

Example 3.1.1 (Eschenburg Spaces). One of the first examples of biquotients comes from the work of Eschenburg [Esc84], [Esc82] and forms a family of spaces called the Eschenburg spaces. An Eschenburg space is a biquotient of the form $E^7_{p,q} = SU(3)//S^1_{p,q}$, with $p, q \in \mathbb{Z}^3$, where the action of $S^1_{p,q}$ on $SU(3)$ is given by

$$z \star A = \operatorname{diag}(z^{p_1}, z^{p_2}, z^{p_3})^{-1} \cdot A \cdot \operatorname{diag}(z^{q_1}, z^{q_2}, z^{q_3}).$$

In order for the quotient to be a manifold, we must impose the condition that $\gcd(p_1 - q_{\sigma(1)}, p_2 - q_{\sigma(2)}) = 1$ for all $\sigma \in S_3$. Alternatievly, if we wish to allow orbifolds, the condition is $\gcd(p_1 - q_{\sigma(1)}, p_2 - q_{\sigma(2)}) \neq 0$.

Eschenburg showed that if $q_i \notin [p_{min}, p_{max}]$, then the Eschenburg space admits a metric with positive curvature, since the argument is done on the Lie algebra level, the same condition holds for orbifolds. Kerin [Ker08] further showed that all Eschenburg spaces admit metrics of quasi-positive curvature, and some admit metrics of almost-positive curvature.

Example 3.1.2 (Gromoll-Meyer Sphere). While systematic study of biquotients began with Eschenburg, an even earlier example of a biquotient is found in [GM74] and is known as the Gromoll-Meyer sphere. The Gromoll-Meyer sphere is the $(2, -1)$ Milnor 7-sphere [Mil56], and is obtained as a biquotient of the form $Sp(2)//Sp(1)$. The embedding of $Sp(1)$ into $Sp(2) \times Sp(2)$ used here is diagonal in the first $Sp(2)$ and the upper-left embedding into the second $Sp(2)$.

Eschenburg and Kerin [EK08] have shown that the Gromoll-Meyer sphere admits a metric with almost positive curvature. More recently, Petersen and Wilhelm [PW08] have proposed a method to show that the Gromoll-Meyer sphere admits a metric with positive curvature.

3.1.2 Orbifold Examples

In addition to the manifold examples presented in the previous section, several example of orbifold biquotients can be found in the literature.

Example 3.1.3. In [FZ07], Florit and Ziller generalized the notion of Eschenburg spaces to a family of 6-dimensional orbifolds of the form $SU(3)//T^2$, where $T^2 = S^1_{p,q} \times S^1_{a,b}$, with both $S^1_{p,q}$ and $S^1_{a,b}$ being of the form required for the Eschenburg space. We study this family in more detail in section 5.3.

Example 3.1.4. In his thesis [Ker08], Kerin studied the family $S^3 \times S^3//T^2$. Kerin showed that there are two families and one exceptional examples of such biquotients. These are described by considering the embedding of T^2 into $T^2 \times T^2 \subset G \times G$, where $G = S^3 \times S^3$.

$$U_L = \left\{ \left(\begin{pmatrix} z \\ w \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) \middle| z, w \in S^1 \right\}$$

$$U_c = \left\{ \left(\begin{pmatrix} z \\ z^c \end{pmatrix}, \begin{pmatrix} w \\ w \end{pmatrix} \right) \middle| z, w \in S^1 \right\} \qquad c \in \mathbb{Z}$$

$$U_{a,b} = \left\{ \left(\begin{pmatrix} z \\ w \end{pmatrix}, \begin{pmatrix} z^a w^b \\ z^a w^b \end{pmatrix} \right) \middle| z, w \in S^1 \right\} \qquad a, b \in \mathbb{Z}$$

$G//U_L = S^2 \times S^2$ and $G//U_0 = \mathbb{CP}^2 \# -\mathbb{CP}^2$ are the only manifolds in this family of biquotients. Moreover, when applying a certain deformation, Kerin shows that $G//T^2$ inherits a metric of almost positive curvature iff it is a singular orbifold.

3.2 Submersions and Deformations

The main reason biquotients have well-behaved curvature properties is that they are formed as base spaces of Riemannian submersions. Recall that O'Neils's formula states that if $M^n \to N^{n-k}$ is a Riemannian submersion, then

$$\sec_N(X, Y) = \sec_M(\widetilde{X}, \widetilde{Y}) + \frac{3}{4} \left| [\widetilde{X}, \widetilde{Y}] \right|^2 \geq \sec_M(\widetilde{X}, \widetilde{Y}),$$

where $\sec_N, \sec_M$ are the sectional curvatures in N and M respectively, and $\widetilde{X}, \widetilde{Y}$ are horizontal lifts of tangent vectors X, Y. In particular, note that Riemannian submersions do not decrease sectional curvature. And so from the submersion $G \to G//U$, we conclude that $G//U$ has non-negative curvature whenever G is a compact Lie group equipped with a bi-invariant metric.

On the other hand, sometimes having non-negative curvature is not the best we can achieve. One way to increase the amount of positive curvature is to perform a

Cheeger deformation [Che73], which is used to improve the curvature on manifolds and orbifolds obtained as quotients of Lie groups.

To perform a Cheeger deformation along a subgroup $K \subset G$, choose $\lambda > 0$, and define $(G, g_\lambda) = G \times_K \lambda K$, where G is equipped with a bi-invariant metric and λK is equipped with the induced metric scaled by λ. g_λ is still left-invariant, but is no longer bi-invariant; however, it is right K-invariant. In particular, we have

Lemma 3.2.1 (Eschenburg). *If (G, K) is a compact symmetric pair, that is there exists an involution $\varphi : \mathfrak{g} \to \mathfrak{g}$ with $Fix(\varphi) = \mathfrak{k}$, equip G with the metric induced by the submersion $G \times \lambda K \to G$, given by $(g, k) \mapsto gk^{-1}$. The metric has non-negative sectional curvature, and* $\sec(X, Y) = 0$ *iff* $[X, Y] = [X^{\mathfrak{k}}, Y^{\mathfrak{k}}] = 0$.

In [Wil02], Wilking used Cheeger deformation to construct a metric of almost positive curvature on $\mathbb{RP}^2 \times \mathbb{RP}^3$ (among other spaces), which disproved the deformation conjecture. Furthermore, a metric of almost positive curvature on $\mathbb{RP}^2 \times \mathbb{RP}^3$ induces a metric of almost positive curvature on $S^2 \times S^3$.

The metric of $S^2 \times S^3$ is achieved by considering the biquotient $S^2 \times S^3 = \Delta(S^3 \times S^3) \backslash (S^3 \times S^3) \times (S^3 \times S^3)/(1 \times \Delta S^1)$, with the metric coming from $S^3 \times S^3$ Cheeger deformed along ΔS^3.

Additionally, Wilking proved a very useful result for studying the points with zero-curvature planes.

Proposition 3.2.2. *(Wilking [Wil02]) Let $M = G//U$ be a normal biquotient, that is a biquotient equipped with a metric induced by the bi-invariant metric on*

G. Suppose $\sigma \subset T_pM$ is a plane satisfying $\sec(\sigma) = 0$. Then the map $\exp : \sigma \to M, v \mapsto \exp(v)$ is a totally geodesic isometric immersion.

While his original result if for normal biquotients, it holds for biquotients with metrics induced by Cheeger deformations as well. In particular, if we have $G//U$, and the metric is deformed along $K \subset G$, then $G//U = (G \times K)//U'$, where $U' = \{((u_l, u_r^{-1}), (k, k)) | (u_l, u_r) \in U, k \in K\} \subset (G \times K) \times (G \times K)$.

Chapter 4

4-Dimensional S^1-Orbifolds

4.1 Examples

In this section we provide some examples of 4-dimensional orbifolds with isometric S^1 actions. We also see that there can be many such orbifolds with the same underlying space, but different singular structures.

Example 4.1.1 (Weighted Projective Spaces). Let $\lambda_0, \lambda_1, \lambda_2$ be positive integers such that

$$\gcd(\lambda_0, \lambda_1, \lambda_2) = 1.$$

We define an S^1 action on $S^5 \subset \mathbb{C}^3$ by

$$z \star (w_0, w_1, w_2) = (z^{\lambda_0} w_0, z^{\lambda_1} w_1, z^{\lambda_2} w_2).$$

The quotient space S^5/S^1 is a 4-dimensional orbifold, known as a weighted projective space, denoted $\mathbb{CP}^2[\lambda_0, \lambda_1, \lambda_2]$, or $\mathbb{CP}^2[\lambda]$ for short.

As with smooth projective spaces, we use homogeneous coordinates, i.e $[w_0 : w_1 : w_2]$ denotes the orbit of (w_0, w_1, w_2).

A typical question when studying orbifolds is: what is the orbifold structure of this space? i.e. what are the singular points, and what are the corresponding orbifold groups?

For $\mathbb{CP}^2[\lambda]$, the following are all the possible non-trivial orbifold groups:

$$\Gamma_{[1:0:0]} = \mathbb{Z}_{\lambda_0} \qquad \Gamma_{[0:1:0]} = \mathbb{Z}_{\lambda_1} \qquad \Gamma_{[0:0:1]} = \mathbb{Z}_{\lambda_2}$$

$$\Gamma_{[w_0:w_1:0]} = \mathbb{Z}_{(\lambda_0,\lambda_1)} \qquad \Gamma_{[w_0:0:w_2]} = \mathbb{Z}_{(\lambda_0,\lambda_2)} \qquad \Gamma_{[0:w_1:w_2]} = \mathbb{Z}_{(\lambda_1,\lambda_2)}.$$

As we can see, $\mathbb{CP}^2[\lambda]$ has a singular set consisting of up to three points corresponding to $[1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1]$ and of up to three (possibly singular) S^2's connecting pairs of such points, which correspond to $[w_0 : w_1 : 0], [w_0 : 0 : w_2], [0 : w_1 : w_2]$.

The metric on $\mathbb{CP}^2[\lambda]$ induced by the round metric on S^5 has positive sectional curvature. Furthermore, the natural action by T^3

$$(z_0, z_1, z_2) \star [w_0 : w_1 : w_2] = [z_0 w_0 : z_1 w_1 : z_2 w_2]$$

has ineffective kernel $S^1 = \{(z^{\lambda_0}, z^{\lambda_1}, z^{\lambda_2})\}$, and hence induces an isometric T^2 action on $\mathbb{CP}^2[\lambda]$.

One can now ask what the stratification of $\mathbb{CP}^2[\lambda]$ is that is induced by an S^1 action, where $S^1 \subset T^2$. The fixed point set of an S^1 action on $\mathbb{CP}^2[\lambda]$ can be either three isolated points, or an isolated point and a (possibly singular) S^2.

The former corresponds to a generic S^1 action, and the fixed points are precisely $[1:0:0], [0:1:0], [0:0:1]$. The latter case has $[1:0:0], [0:w_1:w_2]$ (or similar pairs) as its fixed point set, and corresponds to S^1 actions that can be written as $(z, 1, 1) \in T^3$.

We finish this example by observing that since $\mathbb{CP}^2[\lambda] = S^5/S^1$, the exact homotopy sequence for orbifolds fibrations ($\pi_1^{orb}(E) \to \pi_1^{orb}(B) \to \pi_0(F) \to \pi_0(E) \to \pi_0(B) \to 0$, which one obtains by pulling back the bundle $F \to E \to B$ to a bundle with base $\overline{B}$) implies that $\pi_1^{orb}(\mathbb{CP}^2[\lambda]) = 0$. Also, using Mayer-Vietoris in 3 ways along $[0.5:u:v], [u:0.5:v]$ and $[u:v:0.5]$, one can show that $H^*(|\mathbb{CP}^2[\lambda]|; \mathbb{Z}) = H^*(\mathbb{CP}^2; \mathbb{Z})$. Furthermore, $|\mathbb{CP}^2[\lambda]| = \mathbb{CP}^2$ iff $\lambda_0 = ab, \lambda_1 = ac, \lambda_2 = bc$. This follows from Proposition 2.3.1, by considering the orbifold structure at $[1:0:0], [0:1:0]$, and $[0:0:1]$, since the orbifold group at these points can be generated by elements with non-trivial fixed point sets, implying that $\partial B_\varepsilon = S^3$.

Example 4.1.2. Consider $\mathcal{O} = \mathbb{CP}^2[1, 2, 4]$. and let S^1 act on it by

$$z \star [w_0 : w_1 : w_2] = [zw_0 : w_1 : w_2] = [w_0 : \overline{z}^2 w_1 : \overline{z}^4 w_2],$$

which has ineffective kernel $z = \pm 1$.

The fixed point set consists of an isolated point: $[1:0:0]$, and a singular S^2: $\{[0:w_1:w_2]\}$. To clearly see the representation of S^1 on a neighborhood of $[1:0:0]$, we re-write this action in an effective way as

$$u \star [w_0 : w_1 : w_2] = [w_0 : uw_1 : u^2 w_2],$$

where one can think of u as $\overline{z}^2$. Since the tangent space at $[1:0:0]$ is spanned by $(0,z,w)$, we observe that the action of S^1 on this space has the isotropy representation equivalent to $\varphi_{1,2}$, where $\varphi_{k,l}$ is the action of S^1 on $\mathbb{C}^2=\mathbb{R}^4$ given by $S^1=\{(z^k,z^l)\}\subset T^2$.

This example demonstrates something that can not happen in the manifold case, since Hsiang and Kleiner (Lemma 5 in [HK89]) showed that if the fixed point set contains an isolated point and a 2-dimensional component, then if sec >0, the isotropy representation of S^1 on a neighborhood of the isolated point has to be $\varphi_{1,1}$. In particular, in this case the proof of [HK89] can not immediately be generalized to orbifolds.

Example 4.1.3. Let $\mathcal{O}=S^4/\mathbb{Z}_2$, where we view $S^4\subset\mathbb{C}^2\oplus\mathbb{R}$, and $\mathbb{Z}_2$ acts as $(-1,-1;1)$ on $\mathbb{C}^2\oplus\mathbb{R}$. Thus $\mathcal{O}$ is the suspension of $\mathbb{RP}^3$. Given a point $p\in\mathcal{O}$, we write it as $(\pm(z,w);r)$, where $(z,w;r)\in S^4$ maps to p.

We now introduce an action of $\mathbb{Z}_2$ on $\mathcal{O}$. Let x be the generator of $\mathbb{Z}_2$, then $x\star(\pm(z,w);r)=(\pm(-z,w);r)$. In particular, x fixes the suspension points $(\pm(0,0);1)$ and $(\pm(0,0);-1)$.

If $p=(\pm(0,0);1)$, we split the tangent cone $T_p\mathcal{O}$ into two subspacesV,W as follows: V the projection of the 2-plane $\{(z,0;0)\}\subset T_{(0,0;1)}S^4$, and W the projection of $\{(0,w;0)\}\subset T_{(0,0;1)}S^4$.

Observe that we have $x\star d\varphi(z,0;0)=d\varphi(-z,0;0)=d\varphi(z,0;0)$, and $x\star d\varphi(0,w;0)=d\varphi(0,w;0)$ where $d\varphi:T_{(0,0;1)}S^4\to T_p\mathcal{O}$. In particular, the action of x fixes both V and W. However, x does not fix all of $T_p\mathcal{O}$.

The idea behind this construction is that G acts on $T_{\tilde{p}}\widetilde{U}$ (the tangent space in the local cover) by φ_V, φ_W on lifts of V and W respectively, where $\varphi_V(g), \varphi_W(g) \in \Gamma_p$ for all $g \in G$, but the two are distinct, which implies that $(\varphi_V + \varphi_W)(g) \notin \Gamma_p$ as an action on $T_{\tilde{p}}\widetilde{U}$.

Example 4.1.4. Another interesting family of examples are the Hitchin family of orbifolds introduced in [Hit96]. Recall that a Hitchin orbifold, which we will denote H_k, has S^4 as its underlying space, and its singular locus consists of a smooth Veronese $\mathbb{RP}^2$ with a $\mathbb{Z}_k$ orbifold group. In particular, we view S^4 as the set of traceless symmetric 3x3 matrices with unit norm, on which $SO(3)$ acts by conjugation. We can view each orbit as the space of matrices with fixed eigenvalues.

The singular orbits of the $SO(3)$ action are precisely two copies of $\mathbb{RP}^2$, corresponding to the matrices with repeated positive or negative eigenvalues. To construct the Hitchin k-orbifold H_k, we introduce a $\mathbb{Z}_k$ singularity along one of the $\mathbb{RP}^2$ orbits. One way of interpreting this is to replace the existing D^2 bundle over $\mathbb{RP}^2$ by a $D^2/\mathbb{Z}_k$ cone-bundle over $\mathbb{RP}^2$.

Next, consider the action of $SO(3)$ on $\mathbb{CP}^2$ induced by the canonical embedding $SO(3) \subset SU(3)$. Recall that there exists a branched cover $\mathbb{CP}^2 \to S^4$ where we identify $[w_0 : w_1 : w_2]$ with $[\overline{w_0} : \overline{w_1}, \overline{w_2}]$, and this cover is an $SO(3)$-equivariant continuous map. If we impose a $\mathbb{Z}_k$ singularity along $\mathbb{RP}^2$, we obtain the universal cover of H_{2k} $(\overline{H_{2k}})$.

The Hitchin metric on H_k is self-dual Einstein, but has some negative curvature unless $k = 1, 2$, see [Zil09], where H_1 is the standard S^4, and $H_2 = \mathbb{CP}^2/\mathbb{Z}_2$, where $\mathbb{Z}_2$ acts by conjugation. Furthemore, one can view $H_3 = S^7/SU(2)$, where $SU(2)$ acts by the irreducible representation on $\mathbb{C}^4 \supset S^7$ (see Section 2.2.1).

We note that a Hitchin orbifold has $\pi_1^{orb}(H_k) = 0$ iff k is odd; consider a loop γ corresponding to winding once around the singularity, then $[\gamma] \in \pi_1^{orb}(H_k)$ must have odd order, but if we push it to a non-singular orbit, it has order 1, 2 or 4. Therefore, we must have $[\gamma] = e$, but such loops are the only loops which may be non-trivial. Thus, $\pi_1^{orb}(H_k) = 0$ for k odd. When k is even,H_k is double covered by $\mathbb{CP}^2$ with a singular $\mathbb{RP}^2$ where the orbifold group is $\mathbb{Z}_{k/2}$, with the cover given by the map $\mathbb{CP}^2 \to S^4$ given by identifying $[z]$ with $[\overline{z}]$, where the branching locus is $\mathbb{RP}^2$.

In our context, Hitchin orbifolds are of interest in particular because the two infinite families of 7-dimensional candidates for cohomogeneity one manifolds with positive curvature (P_k, Q_k) can be described as bundles over the Hitchin orbifolds up to covers; namely, $S^3 \to P_k \to H_{2k-1}$ and $S^3 \to Q_k \to \overline{H_{2k}}$ (see [GWZ08] for the general construction, [GVZ11], [Dea11] for positive curvature on P_2, and [Zil09] for an overview). It is conjectured that all manifolds P_k, Q_k admit positive curvature (see [Zil07]). Also, it is known that all P_k, Q_k admit non-negative sectional curvature, hence so do all H_k.

We will now show that one can easily construct a metric of positive curvature on

H_k for all k, as has been observed in [GVZ11] using the work of Müter [Müt87]. Let γ be a geodesic orthogonal to all the $SO(3)$ orbits. The metric on H_k can be defined by considering $v_i(t) = |X_i^*|^2$ at $\gamma(t)$. We begin with a C^∞ function $v : [0, \pi] \to \mathbb{R}_{\geq 0}$ satisfying the following conditions:

$$
\begin{array}{lllll}
v(0) = 0 & v'(0) = 4 & v''(0) = 0 & v'(\pi/3) = 0 & \\
v(\pi) = 0 & v'(\pi) = -4/k & v''(\pi) = 0 & v''(t) < 0 & t \in (0, \pi),
\end{array}
$$

where k is the order of singularity along $\mathbb{RP}^2$. We now split v into three functions $v_1, v_2, v_3 : [0, \pi/3] \to \mathbb{R}_{\geq 0}$, where $v_1(t) = v(t), v_2(t) = v(2\pi/3 + t), v_3(t) = v(2\pi/3 - t)$. These functions define a metric on H_k, where we take X_1, X_2, X_3 as basis for $\mathfrak{so}(3)$ and v_i define the norm of the action field corresponding to X_i at $\gamma(t) \in H_k$, along a geodesic orthogonal to the orbits. These functions satisfy the requirements in Theorem 2.4 in [GVZ11] with $L = \pi/3$ as in [Zil09], and hence the metric is C^2 (and can be made C^∞ as well.) We now observe that given a vector tangent to an orbit, and a vector normal to the orbit, the curvature is given by $\langle R(X_k, T)X_k, T\rangle = -\dfrac{v_k''}{v_k}$ (see [GZ02]), and is hence positive. Without further constraints on the functions v_k, we might have some negative curvature planes. We can now use a Cheeger deformation (see [Che73], [Zil07]) along the action of $SO(3)$ to avoid this problem.

Recall that a Cheeger deformation of a manifold M along the action of a group G is achieved by viewing $M = M \times_G \frac{1}{t}G$, where t is a scaling parameter (see [Müt87]). Given a point $p \in M$, we can write every vector X as $X^N + X^G$, where X^G is tangent to the orbit $G(p)$ and X^N is orthogonal to it. Also, every vector X lifts horizontally to $M \times \frac{1}{t}G$ as

$$(X^N + P^{-1}(P^{-1}+tI)^{-1}X^G, -t(P^{-1}+tI)^{-1}\widetilde{X^G}),$$

where $\widetilde{X^G}$ is an element of $\mathfrak{g}$ whose image under the action is X^G. Furthermore, P is a symmetric 3x3 matrix such that the metric on M is of the form $Q(PX, Y)$ where Q is a bi-invariant metric on G (see [Zil07]).

If we can write $\sigma = \text{span}\{X^N, Y^G\}$, then we have positive curvature. So, we have to consider the case where X^G, Y^G are linearly independent, and without loss of generality, we may assume $Y^N = 0$. We adopt the notation $\mathcal{R}(X,Y) = \langle R(X,Y)X, Y\rangle$ for brevity. By O'Neil's formula we have that

$$\begin{aligned}
\mathcal{R}(X,Y) \geq{}& \mathcal{R}_M(X^N + P^{-1}(P^{-1}+tI)^{-1}X^G, P^{-1}(P^{-1}+tI)^{-1}Y^G) \\
&+ \mathcal{R}_{(1/t)G}(t(P^{-1}+tI)^{-1}\widetilde{X^G}, t(P^{-1}+tI)^{-1}\widetilde{Y^G}) \\
={}& \mathcal{R}_M(X^N, P^{-1}(P^{-1}+tI)^{-1}Y^G) \\
&+ \mathcal{R}_M(P^{-1}(P^{-1}+tI)^{-1}X^G, P^{-1}(P^{-1}+tI)^{-1}Y^G) \\
&+ \mathcal{R}_{(1/t)G}(t(P^{-1}+tI)^{-1}\widetilde{X^G}, t(P^{-1}+tI)^{-1}\widetilde{Y^G}) \\
={}& \mathcal{R}_M(X^N, P^{-1}(P^{-1}+tI)^{-1}Y^G) \\
&+ \mathcal{R}_M(P^{-1}(P^{-1}+tI)^{-1}X^G, P^{-1}(P^{-1}+tI)^{-1}Y^G) \\
&+ t\mathcal{R}_G((P^{-1}+tI)^{-1}\widetilde{X^G}, (P^{-1}+tI)^{-1}\widetilde{Y^G}).
\end{aligned}$$

We now use the fact that the first summand is always positive, the second summand behaves as C_1/t^4 as $t \to \infty$, and the third summand behaves as C_2/t^3 as

$t \to \infty$ with $C_2 > 0$, this can be demonstrated by considering the eigenvalues of P and the eigenvalues of $(P^{-1} + tI)^{-1}$. As such, for t sufficiently large, we get that $\mathcal{R}(X, Y) > 0$ everywhere.

The action by $G = SO(3)$ on $M \times G$ by right multiplication is isometric and induces isometries on H_k. Thus, a circle $S^1 \subset SO(3)$ still acts by isometries. Specifically let

$$S^1 = \left\{ \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix} : t \in [0, 2\pi) \right\} \subset SO(3).$$

Since the $SO(3)$ action on each singular orbit is the standard $SO(3)$ action on $\mathbb{RP}^2$, the S^1 fixes two points, one in each of the singular orbits of the $SO(3)$ action. In particular, it fixes

$$\begin{pmatrix} \frac{1}{\sqrt{6}} & & \\ & \frac{1}{\sqrt{6}} & \\ & & \frac{-2}{\sqrt{6}} \end{pmatrix} \; and \; \begin{pmatrix} \frac{-1}{\sqrt{6}} & & \\ & \frac{-1}{\sqrt{6}} & \\ & & \frac{2}{\sqrt{6}} \end{pmatrix}.$$

We can view the S^1 action as a suspension of an S^1 action on S^3. Indeed, if we view traceless 3x3 symmetric matrices in S^4 as

$$\begin{pmatrix} A & v \\ v^T & h \end{pmatrix} \qquad \text{with } A = \begin{pmatrix} -h/2 + t & b \\ b & -h/2 - t \end{pmatrix}, \qquad \mathrm{tr}A + h = 0.$$

Here $v = \begin{pmatrix} c \\ d \end{pmatrix}$ is a vector in $\mathbb{R}^2$, and h is the suspension parameter. Observe that

$$t^2 + b^2 + c^2 + d^2 = \frac{2 - 3h^2}{4} \qquad \text{and hence } h \in [-2/\sqrt{6}, 2/\sqrt{6}].$$

Thus, we have a 3-sphere when $h \in (-2/\sqrt{6}, 2/\sqrt{6})$, and $t^2 + b^2 + c^2 + d^2 = 0$ when $h = \pm 2/\sqrt{6}$, so the sphere collapses to a point.

Suppose that the singular locus is the $\mathbb{RP}^2$ correpsonding to the matrices with eigenvalues $1/\sqrt{6}$, $1/\sqrt{6}$ and $-2/\sqrt{6}$. Then, conjugating $\text{diag}(1/\sqrt{6}, 1/\sqrt{6}, -2/\sqrt{6})$ we can see that this $\mathbb{RP}^2$ intersects only the spheres with $h \in [-2/\sqrt{6}, 1/\sqrt{6}]$. This intersection is precisely one orbit of the S^1 action (which acts as $\varphi_{1,2}$ on the S^3's, which can be seen from the S^1 action on A, v), and as we approach the last S^3 where the intersection is non-empty, this S^1 turns into the singular orbit.

Example 4.1.5. Another example with $\sec > 0$ follows immediately from the Hitchin orbifolds. In particular, consider the canonical $\mathbb{RP}^2 \subset \mathbb{CP}^2$, and impose a $\mathbb{Z}_k$ singularity along it. That is, a neighborhood of $\mathbb{RP}^2$ is a 2-disk bundle over $\mathbb{RP}^2$, replace this bundle by a $D^2/\mathbb{Z}_k$ cone bundle over $\mathbb{RP}^2$. The gluing along the boundary is well defined since $D^2 \setminus \{0\}/\mathbb{Z}_k$ is diffeomorphic to $D^2 \setminus \{0\}$.

By the discussion in the previous example, this orbifold is the double cover of H_{2k}, and so admits a metric of positive curvature.

Example 4.1.6. Another family of cohomogeneity one orbifolds is $\mathcal{V}_k$, where $|\mathcal{V}_k| = \mathbb{CP}^2$, and the singular locus is an S^2 with a $\mathbb{Z}_k$ singularity. As in the previous example consider the cohomogeneity one action of $SO(3)$ on $\mathbb{CP}^2$, the two singular orbits of this action are $\mathbb{RP}^2$ and S^2. Now consider the orbifolds one gets if the singularity is imposed along the S^2 orbit. (We do this as before, by replacing a disk bundle by a cone bundle.)

Computation shows that $\pi_1^{orb}(\mathcal{V}_k) = 0$ when k is odd and $\pi_1^{orb}(\mathcal{V}_k) = \mathbb{Z}_2$ when k is even. In particular, this implies that there exists a family of 4-dimensional orbifolds $\overline{\mathcal{V}_{2k}}$ which double cover $\mathcal{V}_{2k}$. In fact, this double cover must be a branched double cover, with branching locus the singular S^2. So, $\overline{\mathcal{V}_2}$ is a manifold, and the others have $\overline{\mathcal{V}_2}$ as their underlying space. Since the orbifold singular S^2 is precisely one of the singular orbits of a cohomogeneity one $SO(3)$ action, there must be a cohomogeneity one S^3 action on $\overline{\mathcal{V}_2}$. We claim that $\overline{\mathcal{V}_2} = S^2 \times S^2$.

Consider a cohomogeneity one $SO(3)$ action on $S^2 \times S^2$ given by $A \cdot (u, v) = (Au, Av)$, in the language of [GWZ08], this action has $H = \{e\}, K^{\pm} = SO(2)$ with singular orbits $G/K^+ = \Delta S^2$ and $G/K^- = \{(u, -u) | u \in S^2\}$. Additionally, consider a $\mathbb{Z}_2$ action on $S^2 \times S^2$ given by $x \cdot (u, v) = (v, u)$. Taking the quotient of $S^2 \times S^2$ by this $\mathbb{Z}_2$ action, we get an orbifold with singular locus S^2 (corresponding to $\Delta S^2 \subset S^2 \times S^2$) with a $\mathbb{Z}_2$ singularity. Furthermore, $S^2 \times S^2/\mathbb{Z}_2$ must also have a cohomogeneity one action by $SO(3)$, with $H = S(O(1)O(1)) \cong \mathbb{Z}_2$, $K^+ = SO(2)$ and $K^- = S(O(2)O(1)) \cong O(2)$. We observe that this is precisely the structure of the cohomogeneity one action of $SO(3)$ on $\mathbb{CP}^2$. Therefore, the quotient $S^2 \times S^2/\mathbb{Z}_2$ has $\mathbb{CP}^2$ as its underlying space, with a $\mathbb{Z}_2$ singularity along S^2 corresponding to a singular orbit of the cohomogeneity one $SO(3)$ action. Thus, $S^2 \times S^2/\mathbb{Z}_2$ is precisely $\mathcal{V}_2$.

In particular, $\chi(|\overline{\mathcal{V}_{2k}}|) = 4$, and so by Theorem A, there can be no metric of positive curvature on $\mathcal{V}_{2k}$.

Remark 4.1.1. Note that $\mathcal{O} = \mathbb{CP}^2[1, k, k]$ also has $|\mathcal{O}| = \mathbb{CP}^2$ and an S^2 singular locus with a $\mathbb{Z}_k$ singularity. However, the two orbifolds are distinct, since the resulting loci lie in different π_2 classes. Namely, $\mathbb{CP}^2[1, k, k]^{sing} \sim \pm 1 \in \pi_2(\mathbb{CP}^2)$ and $\mathcal{V}_k^{sing} \sim \pm 2 \in \pi_2(\mathbb{CP}^2)$.

Consider the cohomogeneity one $SO(3)$ action on $\mathbb{CP}^2$, this induces a decomposition of $\mathbb{CP}^2$ into neighborhoods of the two singular orbits $(\mathbb{RP}^2, S^2)$, with intersection $SO(3)/\mathbb{Z}_2 \times I$. This gives us a Mayer-Vietoris sequence

$$H_2(SO(3)/\mathbb{Z}_2) \to H_2(\mathbb{RP}^2) \oplus H_2(S^2) \to H_2(\mathbb{CP}^2) \to$$

$$H_1(SO(3)/\mathbb{Z}_2) \to H_1(\mathbb{RP}^2) \oplus H_1(S^2) \to H_1(\mathbb{CP}^2),$$

plugging in all the groups we have

$$0 \to \mathbb{Z} \to \mathbb{Z} \to$$

$$\mathbb{Z}_4 \to \mathbb{Z}_2 \to 0.$$

This implies that the map $\mathbb{Z} \to \mathbb{Z}$ $(H_2(S^2) \to H_2(\mathbb{CP}^2))$ is multiplication by ± 2. Passing to homotopy by Hurewicz, we conclude that $[\mathcal{V}_k^{sing}] = \pm 2 \in \pi_2(\mathbb{CP}^2)$ (the sign is dictated by the orientation we choose on the singular S^2).

Similarly, the singular locus of $\mathbb{CP}^2[1,k,k]$ is the S^2 singular orbit of the cohomogeneity one $SU(2)$ action on $\mathbb{CP}^2$. Here $H = \{e\}, K^+ = U(1), K^- = SU(2)$. The relevant portion of Mayer-Vietoris is given by

$$H_2(SU(2)) \to H_2(pt) \oplus H_2(S^2) \to H_2(\mathbb{CP}^2) \to$$

$$H_1(SU(2)) \to H_1(pt) \oplus H_1(S^2) \to H_1(\mathbb{CP}^2),$$

plugging in the groups we have

$$0 \to \mathbb{Z} \to \mathbb{Z} \to$$

$$0 \to 0 \to 0.$$

This implies that the map $\mathbb{Z} \to \mathbb{Z}$ $(H_2(S^2) \to H_2(\mathbb{CP}^2))$ is an isomorphism. Passing to homotopy by Hurewicz, we conclude that $[\mathbb{CP}^2[1,k,k]^{sing}] = \pm 1 \in \pi_2(\mathbb{CP}^2)$.

Therefore, there is no homeomorphism $\varphi : \mathbb{CP}^2[1,k,k] \to \mathcal{V}_k$ that maps one singular locus to the other, so the two orbifolds are not diffeomorphic, despite having the same underlying space, homeomorphic singular loci and equal orbifold groups.

4.2 General Structure

In this section, we focus specifically on the structure of 4-dimensional orbifolds with isometric S^1 action. We generally make no assumptions about the curvature.

Lemma 4.2.1. *Let $\mathcal{O}^n$ be a compact Riemannian orbifold with an isometric S^1 action. Let $\mathcal{F}$ be the suborbifold of fixed points of this action. Then,*

1. *Each connected component of $\mathcal{F}$ is a totally geodesic suborbifold of even codimension.*

2. $\chi(|\mathcal{F}|) = \chi(|\mathcal{O}|)$.

Proof. Part 1 is proved completely analogously to the proof for manifold case. (See pp 59-61 of [Kob72].)

For part 2, we note that Kobayashi's original proof in [Kob58] only requires compactness to guarantee that the fixed point set can not be "dense" i.e. there exists $\varepsilon > 0$ such that ε neighborhoods of connected components of the fixed point set are disjoint. This condition is satisfied when we consider S^1 action on strata of a compact orbifold, since such strata are bounded. As such, what we have is $\chi(|\mathcal{F} \cap \mathcal{S}|) = \chi(|\mathcal{S}|)$ for each stratum $\mathcal{S} \subset \mathcal{O}$. Gluing the $\mathcal{F} \cap \mathcal{S}$ pieces together we get part 2. □

Lemma 4.2.2. *Let $\mathcal{O}$ be a compact positively curved 4-dimensional Riemannian orbifold with a non-trivial isometric S^1-action, and $\mathcal{F}$ be the set of fixed points of the action. Then, $\mathcal{F}$ is non-empty and either consists of 2 or more isolated points, or has at least one 2-dimensional component.*

The proof is identical to that for manifolds, the only challenge is to verify that if $\mathcal{F}$ consists of only isolated points, then $|\mathcal{F}| \geq 2$. To do this we utilize Orbifold Poincaré Duality 2.3.8, and Synge's theorem 2.3.6

We also have additional structural restrictions on 4-dimensional orbifolds with an isometric S^1 action.

Proposition 4.2.3. *Let $\mathcal{O}$ be a 4-dimensional Riemannian S^1-orbifold, and let $x \in \mathcal{O}$ be a singular point. Then, we have either*

$$\Gamma_x \subset U(2) \subset SO(4), \qquad \textit{if } x \textit{ is a fixed point, or}$$

$$\Gamma_x \subset SO(3) \subset SO(4), \qquad \textit{otherwise.}$$

Proof. Suppose x is a fixed point. Consider the action of S^1 on $\mathbb{R}^4/\Gamma_x = T_x\mathcal{O}$. Let V denote the vector field associated to the S^1 action on $T_x\mathcal{O}$, and $\widetilde{V}$ its lift to $\mathbb{R}^4$.

$\widetilde{V}$ is a vector field associated to an action of $\mathbb{R}$ on $\mathbb{R}^4$, furthermore, $t_0 = 2\pi$ acts as an element of Γ_x. Therefore, $T = 2\pi n$ acts trivially for some $n \in \mathbb{Z}^+$, so the action is an S^1 action.

Up to conjugation, this S^1 action must be equal to $(e^{lti}, e^{mti}) \in T^2 \subset S^3 \times S^3 = Spin(4)$ (here we consider $Spin(4)$ instead of $SO(4)$ for convenience). Also, this S^1 must normalize Γ_x, and so must commute with Γ_x. This leaves us two cases, either $m \neq 0 \neq l$ or one is zero. If neither m nor l is zero, then Γ_x lifts to T^2, so $\Gamma_x \subset T^2$. Otherwise, the lift of Γ_x is either $S^3 \times S^1$ or $S^1 \times S^3$, in both cases, we get $\Gamma_x \subset U(2) \subset SO(4)$.

Suppose x is not a fixed point, then Γ_x must fix at least one direction (along the orbit $S^1(x)$), so $\Gamma_x \subset SO(3) \subset SO(4)$. □

Since the orbifold group along a stratum must be constant (up to conjugacy), we conclude that

Corollary 4.2.4. *Let $\mathcal{S} \subset \mathcal{O}$ be a stratum that intersects $\mathcal{F}$ but is not contained in it, then $\Gamma_x = \mathbb{Z}_q \subset S^1 = SO(3) \cap T^2 = SU(2) \cap T^2$ for every $x \in \mathcal{S}$.*

4.3 Proof of Theorem A

We consider separately the case where $\mathcal{F}$ is a collection of isolated points, and the case where $\mathcal{F}$ has a 2-dimensional component.

If y is an isolated fixed point, then the lift of the slice representation is equivalent to

$$\varphi_{k,l} : S^1 \times \mathbb{C}^2 \to \mathbb{C}^2; \qquad e^{i\theta} \star (z_1, z_2) = (e^{ik\theta/m} z_1, e^{il\theta/n} z_2),$$

where $k, l \in \mathbb{Z}$ are relatively prime and $(e^{2\pi i/m}, e^{2\pi i/n}) \in \Gamma_y$. Furthermore, $\Gamma_y = \langle (e^{2\pi i/m}, e^{2\pi i/n}) \rangle \oplus \widetilde{\Gamma_y}$. Let $S^3(1) \subset \mathbb{C}^2$ be the unit sphere and let $d : S^3(1) \times S^3(1) \to \mathbb{R}$ be given by $d(v, w) = \angle(v, w)$. Let (X_{kl}, d_{kl}) be the orbit space of $S^3(1)/\widetilde{S^1_{kl}}$ where $\widetilde{S^1_{kl}}$ is a circle that acts by $(e^{ik\theta}, e^{il\theta})$. Furthermore, let $(\widetilde{X_{kl}}, \widetilde{d_{kl}})$ be the quotient of X_{kl} by $\widetilde{\Gamma_y}$.

Lemma 4.3.1. *If $x_1, x_2, x_3 \in \widetilde{X_{kl}}$, then*

$$\widetilde{d_{kl}}(x_1, x_2) + \widetilde{d_{kl}}(x_2, x_3) + \widetilde{d_{kl}}(x_3, x_1) \leq \pi$$

Proof. By Lemma 4 of [HK89], this holds for (X_{kl}, d_{kl}), take lifts of x_i's, apply lemma 4, and then observe that $X_{kl} \to \widetilde{X_{kl}}$ is distance non-increasing. Which gives the desired result. □

We now show that if $\mathcal{F}$ consists of only isolated points, then it has at most 3 points.

Suppose that $\mathcal{F}$ contains at least four points, call them $p_i, 1 \leq i \leq 4$. Let $l_{ij} = \mathrm{dist}(p_i, p_j)$ and let

$$C_{ij} = \{\gamma : [0, l_{ij}] \to \mathcal{O} | \gamma \textit{ length minimizing } p_i \textit{ to } p_j\}.$$

For each triple $1 \leq i, j, k \leq 4$ set

$$\alpha_{ijk} = \min\{\angle(\gamma_j'(0), \gamma_k'(0)) | \gamma_j \in C_{ij}, \gamma_k \in C_{ik}\}.$$

Since $\mathcal{O}$ is compact, the minimum exists.

By Toponogov theorem for orbifolds (see [Sta05]), we get that for i, j, k distinct, $\alpha_{ijk} + \alpha_{kij} + \alpha_{jki} > \pi$. Summing over i, j, k, we get

$$\sum_{i=1}^{4} \sum_{\substack{1 \leq j < k \leq 4 \\ j,k \neq i}} \alpha_{ijk} > 4\pi.$$

On the other hand, by 4.3.1, we know that

$$\sum_{\substack{1 \leq j < k \leq 4 \\ j,k \neq i}} \alpha_{ijk} \leq \pi.$$

Therefore, we can not have more than three isolated fixed points

By Poincaré Duality (Proposition 2.3.8, we conclude that $H^*(|\mathcal{O}|; \mathbb{R})$ is equal to that of either S^4 or $\mathbb{CP}^2$.

For the case when $\dim \mathcal{F} = 2$, we utilize recent work of Harvey and Searle [HS12] on isometries of Alexandrov spaces. In particular, we need the following:

Theorem 4.3.2 ([HS12] Theorem C part (ii))**.** *Let a compact Lie group G act isometrically and fixed-point homogeneously on X^n, a compact n-dimensional Alexandrov space of positive curvature and assume that $X^G \neq \emptyset$ and has a codimension 2 component, then:*

The space X is G-equivariantly homeomorphic to $(\nu * G)/G_p$, where ν is the space of normal directions to $G(p)$ where $G(p)$ is the unique orbit furthest from F.

Remark 4.3.1. The proof of this theorem comes from the work of Perelman on the soul conjecture and Sharafutdinov retraction for Alexandrov spaces. As well as the slice theorem.

Work of Perelman implies that the slice theorem extends to the codimension 2 fixed point set (which we will refer to as N). In the neighborhood of the soul orbit we have $G \times_{G_p} Cone(\nu)$ and in the neighborhood of N we have $Cone(G) \times_{G_p} \nu$. Gluing the two components together we obtain a G-equivariant homeomorphism $|\mathcal{O}| \cong (G * \nu)/G_p$.

Suppose we have $G_p = S^1$ at the soul point, and hence $\nu = S^3/\Gamma$. This implies that $|\mathcal{O}| = (S^3/\Gamma * S^1)/S^1 = (S^5/\Gamma)/S^1 = \mathbb{CP}^2[\lambda]/\widetilde{\Gamma}$.

Taking into account how S^1 must act on $\mathcal{O}$, we conclude that $|\mathcal{O}| = \mathbb{CP}^2[\lambda]/\mathbb{Z}_q$ where $\mathbb{Z}_q \subset T^2 \subset \text{Isom}(\mathbb{CP}^2[\lambda])$, $\mathbb{Z}_q$ fixes $[1:0:0]$, S^1-action lifts to an action on $\mathbb{CP}^2[\lambda]$ with $S^1 \subset T^2$ fixing $[1:0:0]$ and $\{[0:z:w]\}$.

Next suppose that we have $G_p = \mathbb{Z}_q$ at the soul point, and hence $\nu = S^2$. This implies that $|\mathcal{O}| = (S^2 * S^1)/\mathbb{Z}_q = S^4/\mathbb{Z}_q$.

Once again, we use our partial knowledge of the S^1 action to conclude that $|\mathcal{O}| = S^4/\mathbb{Z}_q$ where we view $S^4 \subset \mathbb{C}^2 \oplus \mathbb{R}$, and a generator of $\mathbb{Z}_q$ acts like $x \cdot (z, w; r) = (e^{2\pi i/q} z, e^{2\pi i k/q} w; r)$ where $(k, q) = 1$. The S^1-action lifts to an action on S^4 given by $\theta \cdot (z, w; r) = (e^{i\theta} z, w; r)$.

In particular, the two cases above put together imply Theorem A.

Chapter 5

Orbifold Biquotients of $SU(3)$

5.1 Classification

The first class (the homogeneous spaces) are well known. One simply classifies connected subgroups of $SU(3)$, which up to conjugation are $U(2)$, T^2, $SU(2)$, $SO(3)$ and $S^1_{p,q}$, where $S^1_{p,q} = \mathrm{diag}(z^p, z^q, \overline{z}^{p+q})$ with $p, q \in \mathbb{Z}$. We may assume without loss of generality that $p \geq q \geq 0$. Throughout the rest of this section, we assume that $G//U$ is not given by a homogeneous action, in particular, U must act on both sides.

Recall that the subgroups of $SU(3)$ other than S^1 are unique up to conjugation. For $SU(2)$ and $U(2)$ we will use the standard upper-left block embeddings $\mathrm{diag}(A, 1)$ for $SU(2)$ and $\mathrm{diag}(A, \overline{\det(A)})$ for $U(2)$. For $SO(3)$ we utilize a convenient, although non-standard embedding of $SO(3)$ into $SU(3)$. On the Lie algebra level, we have

$$\mathfrak{so}(3) = \left\{ \begin{pmatrix} ai & 0 & z \\ 0 & -ai & -\overline{z} \\ -\overline{z} & z & 0 \end{pmatrix} \middle| \, a \in \mathbb{R}, z \in \mathbb{C} \right\} \tag{5.1}$$

this embedding of $\mathfrak{so}(3)$ is given by conjugating the standard embedding of $SO(3) \subset SU(3)$ and $\mathfrak{so}(3) \subset \mathfrak{su}(3)$ by

$$g_0 = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & -i \end{pmatrix}. \tag{5.2}$$

The advantage of this embedding is that it has a convenient maximal torus, simplifying some computations. We also note that throughout this paper in the examples which involve $SO(3)$, should the reader desire to utilize the standard $SO(3) \subset SU(3)$, the results stated for $X \in SU(3)$, should now be interpreted as being about $X \cdot g_0$.

From Lemma 3.1.2, we know that if $SU(3)//U$ is an orbifold, then $\operatorname{rk} \mathfrak{u} \leq 2$. In particular, we must have $U = S^1, T^2, SU(2), SO(3), U(2), SU(2) \times S^1$, $SO(3) \times S^1$, a finite quotient of $SU(3), Sp(2)$ or $SU(2) \times SU(2)$, or the exceptional group G_2.

The cases $\mathfrak{u} = \mathfrak{su}(3), \mathfrak{sp}(2), \mathfrak{g}_2$ or $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ can be ruled out quickly. Observe $\dim SU(3)//(SU(3)/\Gamma) = 0$, and in particular the action must be homogeneous. Additionally, $\dim SU(3) < \dim(Sp(2)/\Gamma)$, and $\dim SU(3) < \dim G_2$, so there can be no orbifold biquotients of the form $SU(3)//(Sp(2)/\Gamma)$ or $SU(3)//G_2$. Recall that the only embeddings of $A_1 \oplus A_1 \hookrightarrow \mathfrak{su}(3) \times \mathfrak{su}(3)$ map an A_1 factor into each $\mathfrak{su}(3)$ factor. However, the maximal torii of the possible A_1 embedding are conjugate, which violates the conditions of Lemma 3.1.1.

The cases when U is either S^1 or T^2 yield the 7 and 6 dimensional Eschenburg spaces respectively, and both are covered in section 5.3.

Next, suppose $U = SO(3)$. Since U acts on both sides and there is a unique up to conjugation embedding of $SO(3)$ into $SU(3)$, we must have $U = \Delta SO(3) \subset SU(3) \times SU(3)$. However, this obviously leads to a violation of Lemma 3.1.1. Therefore, there are no non-homogeneous orbifolds of the form $SU(3)//SO(3)$.

Next, suppose $U = SO(3) \times S^1$. Recall that $SO(3) \times S^1$ is not a subgroup of $SU(3)$. Therefore the S^1 and the $SO(3)$ must act on different sides. We get the family of orbifolds $S^1_{p,q}\backslash SU(3)/SO(3)$ whose precise orbifold structure we discuss in section 5.2.1.

Next, we let $U = SU(2)$. By the same argument as for $SO(3)$, we cannot have $U = \Delta SU(2) \subset SU(3) \times SU(3)$. The only remaining non-trivial embedding is if we map U to $SU(2)$ on one side and to $SO(3)$ on the other. We study this embedding on the Lie algebra level. In particular, we use the following bases:

$$\mathfrak{su}(2): \qquad I_1 = \begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$J_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad K_1 = \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\mathfrak{so}(3): \qquad I_2 = \begin{pmatrix} 2i & 0 & 0 \\ 0 & -2i & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{5.3}$$

$$J_2 = \begin{pmatrix} 0 & 0 & \sqrt{2} \\ 0 & 0 & -\sqrt{2} \\ -\sqrt{2} & \sqrt{2} & 0 \end{pmatrix} \quad K_2 = \begin{pmatrix} 0 & 0 & i\sqrt{2} \\ 0 & 0 & i\sqrt{2} \\ i\sqrt{2} & i\sqrt{2} & 0 \end{pmatrix}$$

Under these bases we have: $[I_n, J_n] = 2K_n, [J_n, K_n] = 2I_n, [K_n, I_n] = 2J_n$. We define the embedding $\varphi : \mathfrak{su}(2) \hookrightarrow \mathfrak{su}(3) \oplus \mathfrak{su}(3)$ as $\varphi(I) = (I_1, I_2)$, $\varphi(J) = (J_1, J_2)$ and $\varphi(K) = (K_1, K_2)$. Where I, J, K is the standard quaternionic basis for $\mathfrak{su}(2) = \mathfrak{sp}(1) = \operatorname{Im}\mathbb{H}$. To verify that the resulting biquotient is an orbifold choose $\{tI | t \in \mathbb{R}\}$ as a maximal torus of $\mathfrak{su}(2)$, then $\varphi_1(tI) = tI_1, \varphi_2(tI) = tI_2$. The condition we need to verify is that $tI_1 - \operatorname{Ad}(g)tI_2 = 0$ iff $t = 0$, but $I_2 = 2I_1$, so we have $tI_1 = 2t\operatorname{Ad}(g)I_1$, so $I_1 = 2\operatorname{Ad}(g)I_1$ if $t \neq 0$, but conjugation preserves the norm, so we must have $t = 0$. Therefore the resulting biquotient is an orbifold of dimension 5, which we denote by $\mathcal{O}^5$. For the sake of convenience, we use $SU(2)_\varphi \subset SU(3) \times SU(3)$ to denote this embedding of $SU(2)$, and we write $SU(3)//SU(2)_\varphi$ for the resulting biquotient. We study the orbifold structure of $SU(3)//SU(2)_\varphi$ in section 5.2.1 and its metric properties in section 5.2.2.

The last two cases, $U = SU(2) \times S^1$ and $U = U(2) = (SU(2) \times S^1)/\mathbb{Z}_2$, we consider jointly. The first observation is that in both cases $SU(2) \subset U$. In particular, $SU(3)//SU(2)$ must be an orbifold, where $SU(2) \subset U$. Therefore, $SU(2)$ either acts on only one side, or on both by the above $\varphi : SU(2) \to SU(3) \times SU(3)$. Suppose it is the latter, then there is only one choice of S^1 which commutes with $SU(2)_\varphi$, namely $\mathrm{diag}(z, z, \overline{z}^2)$ acting on the left. We will now show that this does not result in an orbifold.

Consider $\mathrm{diag}(i, i, -2i)$ in the tangent space of the S^1 component, and using the bases in (5.3), (I_1, I_2) in the tangent space of the $SU(2)$ component. For the sum, we have $\mathrm{diag}(2i, 0, -2i)$ on the left, and $\mathrm{diag}(2i, -2i, 0)$ on the right. These two elements of $\mathfrak{su}(3)$ are clearly conjugates, therefore, $SU(3)//(SU(2)_\varphi \times S^1)$ is not an orbifold by Lemma 3.1.1.

Finally, we consider the case where $U = SU(2) \times S^1$ or $U = U(2)$ and $SU(2)$ acts only on one side (we choose the right for convenience). We claim that in this case $SU(3)//U$ is a weighted projective space. Recall that a weighted projective space is defined as $\mathbb{CP}^2[\lambda_0, \lambda_1, \lambda_2] = S^5/S^1$ where the S^1-action is given by

$$w \star (z_0, z_1, z_2) = (w^{\lambda_0} z_0, w^{\lambda_1} z_1, w^{\lambda_2} z_2), \tag{5.4}$$

where $\lambda_i \in \mathbb{Z} \setminus \{0\}$, $\gcd(\lambda_0, \lambda_1, \lambda_2) = 1$ and (z_0, z_1, z_2) are coordinates on $\mathbb{C}^3 \supset S^5$. For $SU(3)/SU(2)$, the possible S^1 biquotient actions are parametrized by $p, q, r \in \mathbb{Z}$, $\gcd(p, q, r) = 1$, none of $p+q, p+r, q+r$ zero, and are induced by the following action on $SU(3)$:

$$z \star X = \mathrm{diag}(z^p, z^q, z^r) X \mathrm{diag}(1, 1, z^{p+q+r})^{-1}, \qquad X \in SU(3).$$

Recall that under our chosen representation of $SU(2) \subset SU(3)$, we have a well-behaved projection map $\pi : SU(3) \to SU(3)/SU(2) = S^5 \subset \mathbb{C}^3$. Namely,

$$\pi \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} = (x_{13}, x_{23}, x_{33}) \in \mathbb{C}^3.$$

Under this identification, the above action becomes

$$z \star (x_{13}, x_{23}, x_{33}) = (\overline{z}^{q+r} x_{13}, \overline{z}^{p+r} x_{23}, \overline{z}^{p+q} x_{33})$$

and hence the quotient is the weighted projective space $\mathbb{CP}^2[-q-r, -p-r, -p-q] \cong \mathbb{CP}^2[q+r, p+r, p+q]$. A note of caution is that this representation need not be in lowest terms, and for proper representation as a weighted $\mathbb{CP}^2$, we need to divide all three weights by their greatest common divisor and normalize the signs to be positive. In general, proper choices of p, q, r allow us to obtain any weighted projective space. As a second note, it does not matter whether the action of $S^1 \times SU(2)$ is effective, so this also covers the case of $SU(3)//U(2)$, which corresponds to the case when p, q, r are all odd.

5.2 New Examples

5.2.1 Orbifold Structure

The singular locus of the generalized Eschenburg spaces will be studied in section 5.3. We will now study the singular locus for the remaining two cases, and start with $\mathcal{O}^5 = SU(3)//SU(2)_\varphi$.

Proposition 5.2.1. *$\mathcal{O}^5 = SU(3)//SU(2)_\varphi$ as defined above has a closed geodesic as its singular locus, and each point on the singular locus has an order 3 orbifold group.*

Proof. Let $\pi : SU(3) \to \mathcal{O}$ denote the projection map. Let φ_1, φ_2 be the projections from $\mathfrak{u}$ onto $\mathfrak{su}(2)$ and $\mathfrak{so}(3)$ respectively, and ψ_1, ψ_2 projections from $U = SU(2)_\varphi \subset SU(3) \times SU(3)$ onto $SU(2)$ and $SO(3)$ respectively. Let $g \in SU(3)$ and $h \in U$ be an element other than identity.

Suppose that $\psi_1(h) \cdot g \cdot \psi_2(h)^{-1} = g$ (i.e. g has non-trivial stabilizer). Since stabilizer groups occur in conjugacy classes, we can assume that h lies in the maximal torus $h = e^{tI} = (e^{tI_1}, e^{tI_2})$ of $SU(2)_\varphi$. Since $\psi_i(e^X) = e^{\varphi_i(X)}$, the condition reduces to

$$g^{-1} \begin{pmatrix} e^{ti} & & \\ & e^{-ti} & \\ & & 1 \end{pmatrix} g = \begin{pmatrix} e^{2ti} & & \\ & e^{-2ti} & \\ & & 1 \end{pmatrix},$$

Since a conjugation can only permute eigenvalues, we see that either $e^{ti} = e^{2ti}$ or $e^{ti} = e^{-2ti}$. The first case is degenerate, since it implies that $e^{ti} = 1$, and so h is identity. In the second case we get that e^{ti} is a third root of unity and

$$g = \begin{pmatrix} & u & \\ v & & \\ & & w \end{pmatrix}, \qquad uvw = -1$$

We now show that $\pi(g)$ lies in a single circle for g as above. Re-write

$$g = \begin{pmatrix} & e^{\lambda i} & \\ -e^{-(\lambda+\mu)i} & & \\ & & e^{\mu i} \end{pmatrix}$$

and act on g by $e^{\lambda/3I} \in U$.

$$g \to \begin{pmatrix} e^{-\lambda i/3} & & \\ & e^{\lambda i/3} & \\ & & 1 \end{pmatrix} \begin{pmatrix} & e^{\lambda i} & \\ -e^{-(\lambda+\mu)i} & & \\ & & e^{\mu i} \end{pmatrix} \begin{pmatrix} e^{2\lambda i/3} & & \\ & e^{-2\lambda i/3} & \\ & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} & 1 & \\ -e^{-\mu i} & & \\ & & e^{\mu i} \end{pmatrix}$$

Define

$$g_z = \begin{pmatrix} & 1 & \\ -\overline{z} & & \\ & & z \end{pmatrix}.$$

So far, we have shown that each singular orbit contains an element of the form g_z. Computations show that $\pi(g_z) = \pi(g_w)$ iff $w = \pm z$. Therefore, we conclude that the image of the singular orbits under π forms a circle. The above also shows that each element in the stabilizer of g_z has order 3, and hence $|\Gamma_{\pi(g_z)}| = 3^n$.

Recall that every group of order p^n where p is prime has a non-trivial center. Additionally, all elements of order 3 in $SU(2)$ are conjugate to each other. Therefore, the stabilizer is abelian. Furthermore, given an element of order 3 inside $SU(2)$, it commutes only with the elements in the same maximal torus. Therefore, the orbifold group is precisely $\mathbb{Z}_3 \subset SU(2)$. □

Remark 5.2.1. We recall that there is a unique smooth 3-dimensional lens space of the form $S^3/\mathbb{Z}^3 = L(3;1) = L(3;2)$. Therefore, it is the space of directions normal to the singular locus.

Next we examine the singular locus of the quotients of the Wu manifold.

Proposition 5.2.2. *The quotient $\mathcal{O}_{p,q} = S^1_{p,q}\backslash SU(3)/SO(3)$ is an orbifold iff $p \geq q > 0$. Furthermore, the action is effective iff $(p,q) = 1$. Its singular locus consists of a singular $\mathbb{RP}^2$ with orbifold group $\mathbb{Z}_2$, with possibly one point on it with a larger orbifold group, and up to two other isolated singular points. The orbifold group at the singular points are $\mathbb{Z}_p, \mathbb{Z}_q$, and $\mathbb{Z}_{p+q}$.*

Proof. To find the singular locus we need to see when $\mathrm{diag}(z^p, z^q, \overline{z}^{p+q})$ is conjugate to an element of $SO(3)$. Without loss of generality, we only need to check when it is conjugate to something in the maximal torus $T \subset SO(3)$. With our choice of $SO(3)$, the most convenient maximal torus has the form $\mathrm{diag}(w, \overline{w}, 1)$

The conjugacy class is determined by the eigenvalues, and hence we must have $\mathrm{diag}(z^p, z^q, \overline{z}^{p+q}) \in S^1_{p,q}$ is conjugate to $\mathrm{diag}(w, \overline{w}, 1) \in SO(3)$ iff z^p, z^q, or $\overline{z}^{p+q}$ is equal to 1. In particular, each of the three choices yields an orbifold group of order p, q, and $p+q$ respectively.

Let $g \in SU(3)$ be a preimage of an orbifold point w.r.t. the action of $S^1_{p,q}$, i.e. $g \cdot \mathrm{diag}(z^p, z^q, \overline{z}^{p+q}) \cdot g^{-1} = \mathrm{diag}(w, \overline{w}, 1)$. We first consider the case where $w^2 \neq 1$. In this case all three eigenvalues are distinct, and so the two diagonal matrices are related by a permutation matrix, i.e. $g = g_i$ as defined below.

$$g_1 = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \qquad g_2 = \begin{pmatrix} & & 1 \\ 1 & & \\ & 1 & \end{pmatrix} \qquad g_3 = \begin{pmatrix} & 1 & \\ & & 1 \\ 1 & & \end{pmatrix}$$

$$g_4 = \begin{pmatrix} & -1 & \\ -1 & & \\ & & -1 \end{pmatrix} \qquad g_5 = \begin{pmatrix} & & -1 \\ & -1 & \\ -1 & & \end{pmatrix} \qquad g_6 = \begin{pmatrix} -1 & & \\ & & -1 \\ & -1 & \end{pmatrix}$$

We note that given g_i, every element of the form $\mathrm{diag}(\rho, \eta, \zeta)X_i$ lies in the same orbit as g_i. For example, for g_1, let $\overline{z}$ be a $(p+q)^{th}$ root of ζ, and $w = \rho\overline{z}^p$. Then, $\mathrm{diag}(z^p, z^q, \overline{z}^{p+q}) \in S^1$, $\mathrm{diag}(w, \overline{w}, 1) \in SO(3)$, and $\mathrm{diag}(z^p, z^q, \overline{z}^{p+q})\mathrm{diag}(w, \overline{w}, 1) = \mathrm{diag}(\rho, \eta, \zeta)$.

Recall that we are using a non-standard $SO(3) \subset SU(3)$, and as such $g_1, g_4 \in SO(3)$, since, $g_4 = g_0 \cdot \mathrm{diag}(-1, 1, -1) \cdot g_0^{-1}$, where g_0 is as in (5.2). This implies that $g_5 = g_2g_4 \in g_2SO(3), g_6 = g_3g_4 \in g_3SO(3)$. Corresponding to three (possibly) isolated singular points.

Now, suppose that $w^2 = 1$ and $w \neq 1$, i.e. $w = -1$. This implies that 2 of $z^p, z^q, \overline{z}^{p+q}$ are -1, and the third is 1. Without loss of generality, assume that $z^p = z^q = -1, \overline{z}^{p+q} = 1$. This implies that either $(p,q) > 1$ or $z = -1$. The former contradicts the assumption that S^1 acts effectively, and hence $z = -1$. Next note that exactly one of $p, q, p+q$ is even. in what follows, we assume that $p+q$ is the even exponent.

We now have $\mathrm{diag}(-1,-1,1) \cdot h \cdot \mathrm{diag}(-1,-1,1) = h$, so h commutes with $\mathrm{diag}(-1,-1,1)$ and so

$$h \in U(2) = \left\{ \begin{pmatrix} A & \\ & \overline{\det(A)} \end{pmatrix} \middle| A \in U(2) \right\} \subset SU(3).$$

To determine $\pi(U(2))$, we need to find the subgroup $K \subset S^1_{p,q} \times SO(3)$ which preserves $U(2)$. Since $S^1 \subset U(2)$, we must have $K = S^1_{p,q} \times (SO(3) \cap U(2))$. The intersection is

$$SO(3) \cap U(2) = \left\{ \begin{pmatrix} w & & \\ & \overline{w} & \\ & & 1 \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} & z & \\ \overline{z} & & \\ & & -1 \end{pmatrix} \right\}.$$

Indeed, for g_0 as in (5.2), we have $g_0 \in U(2)$, and hence $SO(3) \cap U(2) = g_0(SO(3)_{std} \cap U(2))g_0^{-1} = g_0 O(2) g_0^{-1}$, where $O(2) = \{\mathrm{diag}(A, \det(A)) | A \in O(2)\} \subset SU(3)$.

Hence, K is a disjoint union of two copies of T^2. Identifying $U(2) \subset SU(3)$ with the upper 2×2 block, we can rewrite the action of K on $U(2)$. Restricting to the identity component of K, we get

$$(z,w)\star A = \begin{pmatrix} z^p & \\ & z^q \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} w & \\ & \overline{w} \end{pmatrix}^{-1}.$$

Using an appropriate element $z \in S^1_{p,q}$, we can assume that $\det(A) = 1$, and hence $z^{p+q} = 1$. Thus the quotient of this action is the same as $SU(2)//(\mathbb{Z}_{p+q} \times S^1)$, given by

$$(z,w)\star \begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix} = \begin{pmatrix} z^p & \\ & z^q \end{pmatrix} \begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix} \begin{pmatrix} w & \\ & \overline{w} \end{pmatrix}^{-1}$$

where $z^{p+q} = 1$ and $|a|^2 + |b|^2 = 1$. Identifying $SU(2)$ with $S^3 \subset \mathbb{C}^2$ via $\begin{pmatrix} a & -\overline{b} \\ b & -a \end{pmatrix} \mapsto (a,b)$ the S^1 action by w becomes $w \cdot (a,b) \to (aw, bw)$. This is the Hopf action, and hence $S^3/S^1 = S^2$ with projection $S^3 \subset \mathbb{C}^2 \to S^2 = \mathbb{C} \cup \{\infty\}$ given by $(a,b) \mapsto ab^{-1}$. The action by z then becomes $z \cdot (a,b) = (z^p a, z^q b) = (z^p a, \overline{z}^p b)$, since $z^{p+q} = 1$. This induces an action on S^2 given by $ab^{-1} \mapsto z^{2p}(ab^{-1})$. Notice that $z = -1$ acts trivially corresponding to the fact that $\mathbb{Z}_2$ fixed $U(2)$. Thus we have rotation by $2\pi p/(p+q)$, since z runs over the $(p+q)^{th}$ roots of unity.

Finally, we must consider the second component of K, which can be considered as the action on $U(2)$ by $\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$ on the right. On S^3, this action corresponds to $(a,b) \mapsto (\overline{b}, -\overline{a})$, and on $S^2 = \mathbb{C} \cup \{\infty\}$ we get $x \mapsto -1/\overline{x}$, which is precisely the antipodal map. Thus, $\pi(U(2))$ is a (possibly singular) $\mathbb{RP}^2$, and the image of 0 and ∞ is the only orbifold point with orbifold group $\mathbb{Z}_{|p+q|}$. $\square$

Remark 5.2.2. We note that the singular $\mathbb{RP}^2$ above has a distinguished point, which has a larger orbifold group unless the even integer among $p, q, p+q$ is equal to ± 2.

5.2.2 Curvature of $SU(3)//SU(2)$

In this section we study the curvature of the orbifold $SU(3)//SU(2)_\varphi$, and prove Theorem C.

The most natural metric on $\mathcal{O} = SU(3)//SU(2)_\varphi$ is induced by the bi-invariant metric on $SU(3)$. Using this metric, we get

Proposition 5.2.3. *The orbifold $\mathcal{O}^5 = SU(3)//SU(2)_\varphi$, equipped with the metric induced by the bi-invariant metric on $SU(3)$, has quasi-positive curvature.*

Proof. Let $\pi : SU(3) \to \mathcal{O}$ be the projection. We verify that $\mathcal{O}$ has positive sectional curvature at $\pi(I_{SU(3)})$.

The vertical tangent space $T^v_I SU(3) = \mathrm{span}\{I_2 - I_1, J_2 - J_1, K_2 - K_1\}$. The horizontal tangent space $T^h_I SU(3)$ is spanned by

$$X_1 = \begin{pmatrix} i & & \\ & i & \\ & & -2i \end{pmatrix} \qquad X_2 = \begin{pmatrix} & & 1 \\ & & 1 \\ -1 & -1 & \end{pmatrix} \qquad X_3 = \begin{pmatrix} & & i \\ & & -i \\ i & -i & \end{pmatrix}$$

$$X_4 = \begin{pmatrix} & 2\sqrt{2} & 1 \\ -2\sqrt{2} & & -1 \\ -1 & 1 & \end{pmatrix} \qquad X_5 = \begin{pmatrix} & i2\sqrt{2} & i \\ i2\sqrt{2} & & i \\ i & i & \end{pmatrix}$$

We consider two horizontal vectors, $A = \sum a_i X_i, B = \sum b_i X_i$. We want o prove that $[A, B] = 0$ iff A, B are linearly dependent. For convenience, we use $[i, j]$ to denote the quantity $a_i b_j - a_j b_i$. We assume $C = [A, B] = 0$, then we get the following set of identities, where $C_{i,j}$ denotes the (i, j) entry in C.

$$0 = \mathrm{Im}(C_{1,1}) = 2[2,3] + 2[2,5] - 2[3,4] + 18[4,5]$$

$$0 = \mathrm{Re}(C_{1,2}) = 2[2,4] - 2[3,5]$$

$$0 = \mathrm{Im}(C_{1,2}) = 2[2,5] + 2[3,4]$$

$$0 = \mathrm{Re}(C_{1,3}) = -2\sqrt{2}[2,4] - 3[1,3] - 3[1,5] - 2\sqrt{2}[3,5]$$

$$0 = \mathrm{Im}(C_{1,3}) = 3[1,2] + 3[1,4] + 2\sqrt{2}[3,4] - 2\sqrt{2}[2,5] + 4\sqrt{2}[4,5]$$

$$0 = \mathrm{Re}(C_{2,3}) = 3[1,3] - 3[1,5] + 2\sqrt{2}[2,4] + 2\sqrt{2}[3,5]$$

$$0 = \mathrm{Im}(C_{2,3}) = 3[1,2] - 3[1,4] + 2\sqrt{2}[3,4] - 2\sqrt{2}[2,5] - 4\sqrt{2}[4,5]$$

$$0 = \mathrm{Im}(C_{3,3}) = 4[3,4] - 4[2,5].$$

$\mathrm{Im}(C_{1,2}) = 0 = \mathrm{Im}(C_{3,3})$ implies $[3,4] = [2,5] = 0$. Next, $\mathrm{Im}(C_{1,3}) = 0 = \mathrm{Im}(C_{2,3})$ implies $[1,2] = 0$ and $3[1,4] + 4\sqrt{2}[4,5] = 0$. Additionally, $\mathrm{Re}(C_{1,3}) = 0 = \mathrm{Re}(C_{2,3})$ implies $[1,5] = 0$, and $3[1,3] + 2\sqrt{2}[2,4] + 2\sqrt{2}[3,5] = 0$. Furthermore, we get the following relations $[1,3] = \frac{-4\sqrt{2}}{3}[3,5], [1,4] = \frac{-4\sqrt{2}}{3}[4,5], [2,3] = -9[4,5], [2,4] = [3,5]$.

Suppose that $[4,5] \neq 0$, then by scaling we can assume that $[4,5] = 1$. Next, by taking linear combination $A' = \sum a'_i X_i = b_5 A - a_5 B$ and $B' = \sum b'_i X_i = a_4 B - b_4 A$. Observe that $\text{span}\{A, B\} = \text{span}\{A', B'\}$ and $[A', B'] = [4,5] \cdot [A, B] = 0$. So, a'_i, b'_i satisfy the same equations as a_i, b_i. From the construction of a'_i, b'_i we get $a'_2 = a_2 b_5 - a_5 b_2 = [2,5] = 0$. We also get $b'_2 = a_4 b_2 - b_4 a_2 = -[2,4] = -[3,5]$ and $a'_3 = b_5 a_3 - a_5 b_3 = [3,5]$. By a similar construction $a'_4 = b'_5 = 1$ and $a'_5 = b'_4 = 0$. This implies that $a'_2 b'_3 - a'_3 b'_2 = [3,5]^2$, on the other hand, we know $a'_2 b'_3 - a'_3 b'_2 = -9(a'_4 b'_5 - a'_5 b'_4) = -9$, so $[3,5]^2 = -9$ which is impossible. Therefore, we conclude that $[4,5] = 0$, which implies that $[1,4] = [2,3] = 0$.

The remaining possibility is that $[3,5] \neq 0$. As before, assume $[3,5] = 1$, and $A' = b_5 A - a_5 B,\ B' = a_3 B - b_3 A$. The same argument as before yields $a'_i = [i,5]$ and $b'_i = [3,i] = -[i,3]$. Which in particular yields $a'_2 = a'_4 = a'_5 = b'_2 = b'_3 = b'_4 = 0$, $a'_3 = b'_5 = 0$. We conclude that $a'_2 b'_4 - a'_4 b'_2 = 0$, but also $a'_2 b'_4 - a'_4 b'_2 = a'_3 b'_5 - a'_5 b'_3 = 1$, so we conclude that $[3,5] = 0$, and therefore, $[i,j] = 0$ for all i, j. Which is precisely the condition for A and B to be colinear.

Therefore, for $A, B \in T_I^h SU(3)$ linearly independent, $|[A, B]| > 0$, and therefore, $sec(d\pi(A), d\pi(B)) > 0$. So, $\mathcal{O}^5$ has positive curvature at $\pi(I)$. $\square$

Further computations show that the obtained metric on $\mathcal{O}^5_\varphi$ has zero curvature planes along the singular locus. To remedy this, we will now show how to improve this metric using a Cheeger deformation. Recall that in the construction of $\mathcal{O}^5$ we use a non-standard $SO(3)$, see (5.1), and we utilize the bases as of $\mathfrak{so}(3)$ and $\mathfrak{su}(2)$

as in (5.3). We now let

$$\mathfrak{k} = \mathfrak{so}(3) \qquad \textit{and} \qquad \mathfrak{h} = \mathfrak{su}(2).$$

We apply a Cheeger deformation along $SO(3) \subset SU(3)$, which results in a left-invariant, right $SO(3)$-invariant metric. As such, the $SU(2)_\varphi$ acts by isometries, so the deformation induces a new metric on $SU(3)//SU(2)_\varphi$.

Theorem 5.2.4. *$\mathcal{O}^5$ with the metric induced by a Cheeger deformation along the subgroup $SO(3) \subset SU(3)$ has the following properties:*

1. *$\mathcal{O}$ has almost positive curvature.*
2. *The set of points with 0-curvature planes forms a totally geodesic flat 2-torus T that is disjoint from the singular locus.*
3. *Each point in T has exactly one 0-curvature plane, and those planes are tangent to T.*

Proof. As in Chapter 3, we denote the bi-invariant metric on $SU(3)$ by $\langle \cdot, \cdot \rangle$ and the Cheeger deformed metric by $\langle \cdot, \cdot \rangle_\lambda$. Left translations are isometric in $\langle \cdot, \cdot \rangle_\lambda$, and hence we can identify tangent vectors at $g \in SU(3)$ with vectors in $\mathfrak{su}(3)$. Under this identification, the vertical space become $\{\psi(C) - \mathrm{Ad}(g^{-1})C \mid C \in \mathfrak{h}\}$, where $\psi : \mathfrak{h} \to \mathfrak{k}$ is defined by $I_1 \mapsto I_2$, $J_1 \mapsto J_2$ and $K_1 \mapsto K_2$. Then a vector $X \in \mathfrak{su}(3)$ is horizontal at g iff $\langle X, \psi(C) - \mathrm{Ad}(g^{-1})C \rangle_\lambda = 0$ for all $C \in \mathfrak{h}$. Also recall that in $\langle \cdot, \cdot \rangle_\lambda$, a plane spanned by $A, B \in \mathfrak{su}(3)$ is flat iff $[A, B] = [A^{\mathfrak{k}}, B^{\mathfrak{k}}] = 0$. Additionally, for some computations in this proof, we recall that $\langle X,Y\rangle_\lambda = \langle X^{\mathfrak{k}^\perp}, Y^{\mathfrak{k}^\perp}\rangle + \frac{\lambda}{1+\lambda}\langle X^{\mathfrak{k}}, Y^{\mathfrak{k}}\rangle$. We will use $\nu = \frac{\lambda}{1+\lambda} \in (0,1)$ for brevity.

Suppose that at the image of some point $g \in SU(3)$, we have 0-curvature. Let A, B be two elements of T_g^h (left translated to the identity), which span a 0-curvature plane.

We begin by making a series of claims:

1. We may assume $A^{\mathfrak{k}} = 0$.

   Indeed, if $A^{\mathfrak{k}} \neq 0$ and $B^{\mathfrak{k}} \neq 0$, then $[A^{\mathfrak{k}}, B^{\mathfrak{k}}] = 0$ iff $A^{\mathfrak{k}} = c \cdot B^{\mathfrak{k}}$ since $\mathfrak{k}$ has rank one.

2. $\langle \mathrm{Ad}(g)A, \mathfrak{h}\rangle = 0$.

   Observe that since $\langle A, \mathfrak{k}\rangle = 0$, we have $\langle \mathrm{Ad}(g)A, \mathfrak{h}\rangle = \langle A, \mathrm{Ad}(g^{-1})\mathfrak{h}\rangle = \langle A, \mathrm{Ad}(g^{-1})\mathfrak{h}\rangle_\lambda$. If $\langle A, \mathrm{Ad}(g^{-1})(C)\rangle_\lambda \neq 0$, then, since $A \in \mathfrak{k}^\perp$, we have $\langle A, \psi(C) - \mathrm{Ad}(g^{-1})C\rangle_\lambda \neq 0$, so A is not horizontal.

3. $B^{\mathfrak{k}} \neq 0$.

   Suppose that $B^{\mathfrak{k}} = 0$, then by the same argument as we used with A, we must have $\langle \mathrm{Ad}(g)B, \mathfrak{h}\rangle = 0$. In particular, both $\mathrm{Ad}(g)A$ and $\mathrm{Ad}(g)B$ are horizontal with respect to the submersion $SU(3) \to S^5 = SU(3)/SU(2)$. Since, endowed with the metric induced by $\langle\cdot,\cdot\rangle$, S^5 has positive sectional curvature, it follows that $0 \neq [\mathrm{Ad}(g)A, \mathrm{Ad}(g)B] = \mathrm{Ad}(g)[A,B]$, in particular, $[A,B] \neq 0$.

4. Up to scaling $A = \mathrm{Ad}(u)X$ for some $u \in SO(3)$, where we take $X = \mathrm{diag}(i, i, -2i)$.

   Observe that $B^{\mathfrak{k}} = t \cdot \mathrm{Ad}(u) I_2$ for some non-zero $t \in \mathbb{R}$ and $u \in SO(3)$. Notice that since $A \in \mathfrak{k}^{\perp}$, it follows that $[A, B^{\mathfrak{k}}] \in \mathfrak{k}^{\perp}$ and since $(\mathfrak{g}, \mathfrak{k})$ is a symmetric pair, $[A, B^{\mathfrak{k}^{\perp}}] \in \mathfrak{k}$, and hence we must have $[A, B^{\mathfrak{k}}] = [A, B]^{\mathfrak{k}^{\perp}} = 0$. Therefore, $0 = [A, B^{\mathfrak{k}}] = t \cdot \mathrm{Ad}(u)[\mathrm{Ad}(u^{-1})A, I_2]$. A matrix that commutes with I_2 must be diagonal. Since $A \in \mathfrak{k}^{\perp}$, and $u \in SO(3)$, $\mathrm{Ad}(u^{-1})A \in \mathfrak{k}^{\perp}$ as well. In particular, $\mathrm{Ad}(u^{-1})A$ is also orthogonal to I_2, which implies our claim. Furthermore, since scaling A does not change the plane spanned by A, B we will assume that $A = \mathrm{Ad}(u)X$.

5. By changing the point in the orbit, we may assume that $g \in U(2)$.

   First, we observe that $\langle \mathrm{Ad}(gu)X, \mathfrak{h} \rangle = \langle \mathrm{Ad}(g)A, \mathfrak{h} \rangle = 0$, since $A = \mathrm{Ad}(u)X$.

   Let

$$gu = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \qquad \text{and} \qquad \mathrm{Ad}(gu)X = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix}.$$

   Since this has to be orthogonal to $\mathfrak{h}$ with respect to the bi-invariant metric, we conclude that $m_{13} = 0, m_{23} = 0, m_{11} = m_{22}$ and $m_{33} = -2m_{11}$. We compute that $m_{13} = (a_1\overline{c_1} + a_2\overline{c_2} - 2a_3\overline{c_3})i = (-3a_3\overline{c_3})i$. This is zero iff $a_3 = 0$ or $b_3 = 0$. Similarly, $m_{23} = 0$ implies $b_3 = 0$ or $c_3 = 0$. We also compute that $m_{11} = (|a_1|^2 + |a_2|^2 - 2|a_3|^2)i = (1 - 3|a_3|^2)i$ and $m_{22} = (|b_1|^2 + |b_2|^2 - 2|b_3|^2)i =$

$(1 - 3|b_3|^2)i$. Since $m_{11} = m_{22}$, we must have $|b_3| = |a_3|$. Finally, we see that $m_{33} + 2m_{11} = (1 - |c_3|^2) + 2(1 - 3|a_3|^2) = 0$. Suppose that $c_3 = 0$, then $|a_3|^2 + |b_3|^2 = 1$, so $|a_3|^2 = 1/2$, but this implies that $1 + (2 - 3 \cdot 1/2) = 3/2 \neq 0$, so $m_{33} \neq -2m_{11}$. Therefore, $c_3 \neq 0$, which implies that both a_3 and b_3 are zero, which also implies $c_1 = c_2 = 0$. Therefore, $gu \in U(2) \subset SU(3)$, and so $g \in SU(2)\mathrm{diag}(w, w, \overline{w}^2)SO(3)$. Thus, $g = u \cdot \mathrm{diag}(w, w, \overline{w}^2) \cdot v$ with $u \in SU(2), v \in SO(3)$. Since there exists $x \in SU(2)$ such that $(x, v) \in SU(2)_\varphi$, we can change the point in the orbit and assume that $g \in U(2)$.

6. We may assume that $A = X = \mathrm{diag}(i, i, -2i)$.

Since $\langle A, \mathrm{Ad}(g^{-1})\mathfrak{h}\rangle = 0$ and $g \in U(2)$, it follows that $A \in \mathfrak{h}^\perp$, and as we saw, $A \in \mathfrak{k}^\perp$ as well. Thus,

$$A \in \mathfrak{k}^\perp \cap \mathfrak{h}^\perp = \left\{ \begin{pmatrix} ti & 0 & z \\ 0 & ti & \overline{z} \\ -\overline{z} & -z & -2ti \end{pmatrix} \middle| \, t \in \mathbb{R}, z \in \mathbb{C} \right\}.$$

We furthermore know that A is conjugate to $\mathrm{diag}(i, i, -2i)$ and so has eigenvalues $i, i, -2i$. In particular, there is a repeated pair. The eigenvalues of A as above are:

$$ti, -ti/2 \pm 1/2\sqrt{-9t^2 - 8|z|^2}$$

The last two are equal iff $t = z = 0$, which means we have $A = 0$. Therefore, the repeated eigenvalue is ti. So, the determinant must be $(ti)^2(-2ti) = 2t^3 i$.

Computing the determinant of A, we have $\det(A) = 2t^3 i + 2t|z|^2 i$. Therefore $z = 0$, and hence $A = X$.

We now examine the possible values of B. Since $[A, B] = 0$, this implies that $B = \begin{pmatrix} ri & z & \\ -\overline{z} & -(r+s)i & \\ & & si \end{pmatrix}$, where $r, s \in \mathbb{R}$, $z \in \mathbb{C}$. We can also assume that $s = 0$ by replacing B with $B + (s/2)A$.

The question now becomes when a B of this form is a horizontal vector at

$$g = \begin{pmatrix} a & b & \\ -\overline{b} & \overline{a} & \\ & & 1 \end{pmatrix} \begin{pmatrix} w & & \\ & w & \\ & & \overline{w}^2 \end{pmatrix}.$$

We begin by noting that we have

$$I_2 - \mathrm{Ad}(g^{-1})I_1 = \begin{pmatrix} (2 + |b|^2 - |a|^2)i & -2i\overline{a}b & \\ -2i\overline{b}a & (-2 + |a|^2 - |b|^2)i & \\ & & 0 \end{pmatrix}$$

$$J_2 - \mathrm{Ad}(g^{-1})J_2 = \begin{pmatrix} -ab + \overline{ab} & -b^2 - \overline{a}^2 & \sqrt{2} \\ a^2 + \overline{b}^2 & ab - \overline{ab} & -\sqrt{2} \\ -\sqrt{2} & \sqrt{2} & 0 \end{pmatrix}$$

$$K_2 - \mathrm{Ad}(g^{-1})K_2 = \begin{pmatrix} (ab + \overline{ab})i & (b^2 - \overline{a}^2)i & \sqrt{2}i \\ (\overline{b}^2 - a^2)i & -(ab + \overline{ab})i & \sqrt{2}i \\ \sqrt{2}i & \sqrt{2}i & 0 \end{pmatrix}$$

as a basis for the vertical tangent space.

From $\langle B, I_2 - \mathrm{Ad}(g^{-1})I_1\rangle_\lambda = 0$, we get:

$$\begin{aligned}
0 &= \langle B, I_2 - \mathrm{Ad}(g^{-1})I_1\rangle_\lambda \\
&= \nu r(2 + |b|^2 - |a|^2) + (-\overline{a}\overline{z}bi + az\overline{b}i) \\
&= \nu r(3|b|^2 + |a|^2) + 2Im(\overline{a}\overline{z}b).
\end{aligned}$$

So, $r = \dfrac{-2Im(\overline{a}\overline{z}b)}{\nu(|a|^2 + 3|b|^2)}$.

If we plug in $a = 0$ or $b = 0$, we get $r = 0$, so $B^{\mathfrak{k}} = 0$, which contradicts one of our earlier observations. So we may assume that $a \neq 0, b \neq 0$. Under these assumptions, plugging in what we obtained for r, we get:

$$\begin{aligned}
0 &= \langle B, J_2 - \mathrm{Ad}(g^{-1})J_1\rangle_\lambda \\
&= \frac{-1}{2(|a|^2 + 3|b|^2)}\Big(3|a|^2b^2\overline{z} + a^2|b|^2z + \overline{a}^2|b|^2\overline{z} + 3|a|^2\overline{b}^2z + 3|b|^2b^2\overline{z} \\
&\quad + |a|^2\overline{a}^2\overline{z} + |a|^2a^2z + 3|b|^2\overline{b}^2z\Big) \\
&= \frac{-1}{2(|a|^2 + 3|b|^2)}(|a|^2 + |b|^2)(a^2z + 3b^2\overline{z} + \overline{a}^2\overline{z} + 3\overline{b}^2z) \\
&= \frac{-1}{|a|^2 + 3|b|^2}Re(a^2z + 3b^2\overline{z}).
\end{aligned}$$

So, $Re(a^2z + 3b^2\overline{z}) = 0$.

Similarly,

$$
\begin{aligned}
0 &= \langle B, K_2 - \mathrm{Ad}(g^{-1})K_1 \rangle_\lambda \\
&= \frac{-i}{2(|a|^2+3|b|^2)}\Big(-3|a|^2 b^2\overline{z} - a^2|b|^2 z + \overline{a}^2|b|^2\overline{z} + 3|a|^2\overline{b}^2 z - 3|b|^2 b^2\overline{z} \\
&\qquad + |a|^2\overline{a}^2\overline{z} - |a|^2 a^2 z + 3|b|^2\overline{b}^2 z\Big) \\
&= \frac{-i(|a|^2+|b|^2)}{2(|a|^2+3|b|^2)}(-3b^2\overline{z} - a^2 z + \overline{a}^2\overline{z} + 3\overline{b}^2 z) \\
&= \frac{1}{|a|^2+3|b|^2} Im(3b^2\overline{z} + a^2 z).
\end{aligned}
$$

So, $Im(a^2z + 3b^2\overline{z}) = 0$.

Together, these observations imply that $a^2z + 3b^2\overline{z} = 0$, so $|a|^2 = 3|b|^2$, hence $|b|^2 = 1/4$ and $|a|^2 = 3/4$. Furthermore, given a and b, z is unique up to scaling (which also scales r). Therefore B, if it exists, is unique up to scaling. This proves that each point at which there exists a plane of zero-curvature has a unique such plane.

Furthermore, we have

$$
g = \begin{pmatrix} \frac{\sqrt{3}}{2}e^{ti} & \frac{1}{2}e^{si} & \\ -\frac{1}{2}e^{-si} & \frac{\sqrt{3}}{2}e^{-ti} & \\ & & 1 \end{pmatrix} \begin{pmatrix} w & & \\ & w & \\ & & \overline{w}^2 \end{pmatrix}.
$$

By applying $(\mathrm{diag}(e^{ti}, e^{-ti}, 1), \mathrm{diag}(e^{2ti}, e^{-2ti}, 1)) \in SU(2)_\varphi$ to g, we see that by changing the point in the orbit, we can assume that

$$g = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2}e^{s'i} & \\ -\frac{1}{2}e^{-s'i} & \frac{\sqrt{3}}{2} & \\ & & 1 \end{pmatrix} \begin{pmatrix} w & & \\ & w & \\ & & \overline{w}^2 \end{pmatrix}.$$

It is easy to verify that the choice of such representative element is unique up to replacing w by $-w$. Therefore, the set of points with 0-curvature planes is in a one-to-one correspondence to a 2-torus.

We now use a result by Wilking:

Proposition 5.2.5. *(Wilking [Wil02]) Let M be a normal biquotient. Suppose $\sigma \subset T_pM$ is a plane satisfying* $\sec(\sigma) = 0$. *Then the map* $\exp : \sigma \to M, v \mapsto \exp(v)$ *is a totally geodesic isometric immersion.*

To apply this to a biquotient $G//U$ with a Cheeger deformed metric, observe that $G//U = (G \times \lambda K)//U'$, where $U' = \{((u_l, u_r^{-1}), (k, k)) | (u_l, u_r) \in U, k \in K\} \subset (G \times K) \times (G \times K)$. In this context, Wilking's result tells us that exponentiating a flat plane results in a flat totally geodesic subspace. Let T be the set of all points in $SU(3)//SU(2)_\varphi$ with flat planes. If $p \in T$ and σ is the unique flat 2-plane at p, it follows that near p we have $T = \exp_p \sigma$. In particular, T is smooth and hence diffeomorphic to a 2-torus. Furthermore, it follows that for all $p \in T$, the unique flat plane must be tangent to T. This concludes the proof of Theorem 5.2.4. □

Theorem C follows immediately from Theorem 5.2.4, and in particular, Theorem 5.2.4 tells us what metric to use for Theorem C. An interesting question is whether the metric in Theorem 5.2.4 can be further deformed to give a metric of positive curvature. The author has made an attempt to achieve this by doing an additional Cheeger deformation along $SU(2)$ on the left; however, the curvature properties appear to be unchanged.

Proof of Corollary 1.0.1. Observe that

$$S^1 = \left\{ \begin{pmatrix} z & & \\ & z & \\ & & \overline{z}^2 \end{pmatrix} \right\}$$

acts on the left on $SU(3)$. Furthermore, this S^1 commutes with the $SU(2)$ action in the construction of $\mathcal{O}$. Therefore, we get an Alexandrov space $X^4 = \mathcal{O}/S^1$.

Furthermore, note that each 0-curvature plane of $\mathcal{O}^5$ contains a direction (the vector A from before) tangent to the fiber of this action. Therefore, by O'Neil's formula, X^4 has positive curvature.

Additionally, note that when $z = -1$, the action is trivial, and corresponds to the action of $-I \in SU(2)_\varphi \subset SU(3) \times SU(3)$. Therefore, we conclude that $X^4 = SU(3)//U(2)$. □

5.3 Generalized Eschenburg Spaces

5.3.1 Seven Dimensional Family

First introduced in [Esc84], Eschenburg spaces are a family of 7-dimensional manifolds (the construction can be generalized to orbifolds as well), that all admit quasi-positive curvature [Ker08], and many of which admit positive curvature. Eschenburg spaces are defined as

$$E^7_{p,q} = SU(3)//S^1_{p,q}$$

where $p, q \in \mathbb{Z}^3$, $\sum p_i = \sum q_i$. Furthermore for the action to be free, we need that

$$(p_1 - q_{\sigma(1)}, p_2 - q_{\sigma(2)}) = 1 \qquad \text{for any } \sigma \in S_3.$$

More generally, if we allow Eschenburg orbifolds, then the condition is relaxed to p and q not being permutations of each other, in other words, for $\sigma \in S_3$ we have

$$(p_1 - q_{\sigma(1)}, p_2 - q_{\sigma(2)}) \neq 0.$$

The action of $S^1_{p,q}$ on $SU(3)$ is given by

$$z \star X = \mathrm{diag}(z^{p_1}, z^{p_2}, z^{p_3}) \cdot X \cdot \mathrm{diag}(\overline{z}^{q_1}, \overline{z}^{q_2}, \overline{z}^{q_3}).$$

Eschenburg showed that this space admits a metric of positive curvature when deformed along one of the three block embeddings of $U(2) \subset SU(3)$, iff $q_i \notin$

$[\min\{p_j\}, \max\{p_j\}]$ for each i. Kerin further showed that all Eschenburg spaces have quasi-positive curvature, and if

$$q_1 < q_2 = p_1 < p_2 \leq p_3 < q_3 \; or \; q_1 < p_1 \leq p_2 < p_3 = q_2 < q_3,$$

the metric has almost positive curvature. Since all the above results are proven on the Lie algebra level, they hold when we generalize to Eschenburg orbifolds.

Before examining the idea of orbifold fibrations of Eschenburg spaces by Florit and Ziller [FZ07], let us first examine the orbifold structure of Eschenburg orbifolds, since it will be similar to that of the orbifold fibrations.

It is easy to verify that the singular locus of an Eschenburg orbifold $SU(3)//S^1_{p,q}$ consists of some combination of circles denoted $\mathcal{C}_\sigma$ and lens spaces (possibly including S^3 and $S^2 \times S^1$) denoted $\mathcal{L}_{ij}$. Furthermore, each such component is a totally geodesic suborbifold. In this construction we include a minor correction to the work of Florit and Ziller to ensure that $U(2)_{ij}$ and T^2_σ are always subsets of $SU(3)$.

We define $\mathcal{L}_{ij}$ to be the images in $E^7_{p,q}$ of $U(2)_{ij} \subset SU(3)$, defined as

$$U(2)_{ij} = \left\{ \tau_i \begin{pmatrix} A & 0 \\ 0 & \overline{\det A} \end{pmatrix} \tau_j : A \in U(2) \right\}, \qquad 1 \leq i, j \leq 3$$

where $\tau_i \in S_3 \subset O(3)$ with τ_1, τ_2 interchanging the 3^{rd} vector with the 1^{st} and 2^{nd} respectively, and $\tau_3 = -I$.

Furthermore, we define $\mathcal{C}_\sigma$ for $\sigma \in S_3$ as the projections of T^2_σ, which are defined as

$$T^2_\sigma = \mathrm{sgn}(\sigma)\sigma^{-1}\mathrm{diag}(z, w, \overline{zw})$$

where we view $S_3 \subset O(3)$ and $\text{sgn}(\sigma)$ is 1 if σ even and -1 if σ odd.

From this listing, we can observe that the singular locus has the following structure

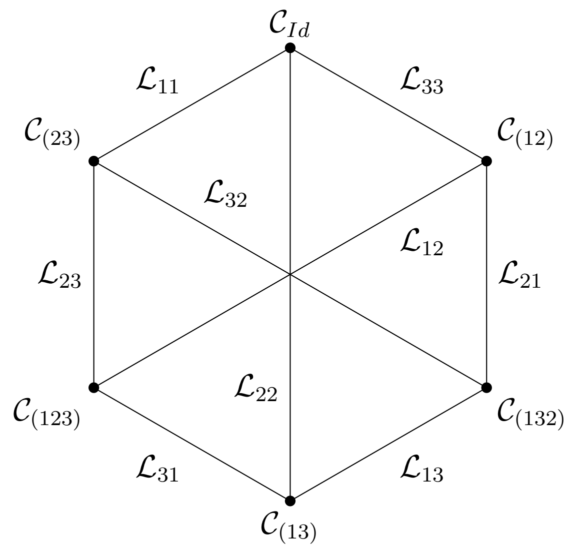

Figure 5.1: Structure of the singular locus [FZ07]

where $\mathcal{L}_{ij}$ connecting $\mathcal{C}_\sigma$ and $\mathcal{C}_\tau$ means that both $\mathcal{C}_\sigma$ and $\mathcal{C}_\tau$ lie in $\mathcal{L}_{ij}$. We compute the orbifold groups along $\mathcal{C}_\sigma$ in Theorem 5.3.6, and the orbifold groups along $\mathcal{L}_{ij}$ are implied by Lemma 5.3.5.

5.3.2 Construction of the Six Dimensional Family

For the construction of the six dimensional family of Exchenburg spaces, Florit and Ziller [FZ07] considered fibrations of the form $E^7_{p,q}/S^1_{a,b}$. Given $a, b \in \mathbb{Z}^3, \sum a_i = \sum b_i$ we define $S^1_{a,b}$ acting on $SU(3)$ as before, and furthermore, this action induces an action of $S^1_{a,b}$ on $E^7_{p,q}$. In this paper we consider these orbifolds more directly as $\mathcal{O}^{a,b}_{p,q} = SU(3)//T^2$, where T^2 is generated by the two circles $S^1_{p,q}$ and $S^1_{a,b}$.

Florit and Ziller prove that

Theorem 5.3.1. *The action of $T^2 = S^1_{a,b} \times S^1_{p,q}$ on $SU(3)$ is almost free iff*

$$(p - q_\sigma) \text{ and } (a - b_\sigma) \text{ are linearly independent}, \ for\ all\ \sigma \in S_3.$$

The quotient $\mathcal{O}^{a,b}_{p,q}$ is then an orbifold whose singular locus is the union of at most nine orbifold 2-spheres and six points that are arranged according to the schematic diagram in Figure 5.1 above.

5.3.3 Equivalence of Actions by T^2

It is clear that $\mathcal{O}^{a,b}_{p,q} = \mathcal{O}^{p,q}_{a,b}$. A natural question is what other ways are there to write the same biquotient?

Proposition 5.3.2. *$\mathcal{O}^{a,b}_{p,q} = \mathcal{O}^{a',b'}_{p',q'}$ whenever $a', b', p', q' \in \mathbb{Z}^3$ are given as follows:*

1. $a' = b, b' = a, p' = q, q' = p$

2. $a' = \lambda a, b' = \lambda b,\ p' = \mu p, q' = \mu q$ *where* $\lambda, \mu \in \mathbb{Q} \setminus \{0\}$.

3. $a' = (a_1 + c, a_2 + c, a_3 + c), b' = (b_1 + c, b_2 + c, b_3 + c),\ p' = (p_1 + d, p_2 + d, p_3 + d), q' = (q_1 + d, q_2 + d, q_3 + d)$ *where* $c, d \in \mathbb{Z}$.

4. $a' = \sigma(a), b' = \tau(b), p' = \sigma(p), q' = \tau(q)$, *where* $\sigma, \tau \in S_3$ *act by permutation.*

5. $$\begin{pmatrix} a' \\ p' \end{pmatrix} = A \begin{pmatrix} a \\ p \end{pmatrix} \qquad \begin{pmatrix} b' \\ q' \end{pmatrix} = A \begin{pmatrix} b \\ q \end{pmatrix}$$

where $A \in GL_2(\mathbb{Z})$.

Proof. The first 4 are simply adaptations of the equivalence rules for Eschenburg spaces. The fifth one is simply a reparametrization of the T^2 corresponding to a change of basis. □

Two important corollaries of this proposition will allow us to only deal with effective actions of T^2.

Corollary 5.3.3. *Given an action of* T^2 *on* $SU(3)$ *with a finite ineffective kernel, the above operations allow us to write the same quotient as* $SU(3)//T^2$ *with an effective action.*

Proof. Let (z_0, w_0) be an element of the ineffective kernel of order n. We can choose integers k, l such that $0 \leq k, l < n$, $\gcd(k, l, n) = 1$, and

$$z_0 = e^{2\pi i k/n}, \qquad w_0 = e^{2\pi i l/n}.$$

We now consider a different generator. Let s be such that $(k, l)s \equiv 1 \pmod{n}$. Consider $(z_1, w_1) = (z_0^s, w_0^s)$, since $(n, s) = 1$, we must have (z_1, w_1) and (z_0, w_0) generate the same subgroup, but furthermore, we have

$$z_1 = e^{2\pi i k'/n}, \qquad w_1 = e^{2\pi i l'/n},$$

where $k' = k/(k, l), l' = l/(k, l)$, and so $(k', l') = 1$.

Next let α, β be integers such that $\alpha l' - \beta k' = 1$. Then, we apply transformation 5 above with

$$A = \begin{pmatrix} l' & k' \\ \beta & \alpha \end{pmatrix}.$$

Under this transformation, (z_1, w_1) gets changed to (u_1, v_1), where $v_1 = 1$, and $u_1 = e^{2\pi i/n}$. Let the action by (u, v) be denoted as

$$(u, v) \cdot X = u^{p'} v a' X \overline{u}^{q'} \overline{v}^{b'}.$$

Then, the fact that $(e^{2\pi i/n}, 1)$ is in the ineffective kernel, means that $p_1 \equiv p_2 \equiv p_3 \equiv q_1 \equiv q_2 \equiv q_3 \pmod n$. Apply transformation 3 above with $c = 0$ and $d = -p_1$, to get that $p_i \equiv q_i \equiv 0 \pmod n$. Next applying transformation 2, with $\lambda = 1/n$, we kill off this generator of the ineffective kernel.

Repeating this process for all generators of the ineffective kernel guarantees that the action is effective. □

For the next corollary, we consider a special subfamily of the seven dimensional Eschenburg spaces: $E_d^7 = E^7_{(1,1,d),(0,0,d+2)}$ for $d \geq 0$ is a family of cohomogeneity one manifolds, which means that there exists a group G which acts on E_d^7 by isometries with $\dim E_d^7/G = 1$. In this case, $G = SO(3)SU(2)$ (see [GWZ08]). For $d > 0$, Eschenburg's construction gives us a metric of positive curvature on E_d^7.

In general, the approach in Corollary 5.3.3 makes no guarantees that the effective T^2 shares a generating circle with the initial T^2 action. The following corollary addresses this shortcoming for the particular case when $SU(3)//S^1_{p,q} = E_d^7$.

Corollary 5.3.4. *If $\mathcal{O}^{a,b}_{p,q} = E^7_d//S^1_{a,b} = SU(3)//T^2$ has ineffective torus action, then we can rewrite it as $E^7_d//S^1_{a',b'}$ with an effective torus action.*

Proof. Let $S^1_{a,b}$ act by $w^{\alpha,\beta,0}$ on the left and $\overline{w}^{\gamma,\delta,\varepsilon}$ on the right. Since ε is uniquely determined by the other 4 indecies, we will mostly ignore it.

The innefective kernel has order given by $k = \gcd(\gamma - \delta, \alpha - \beta, \alpha d - \gamma(d-1))$.

In particular, $\alpha \equiv \beta \pmod k$ and $\gamma \equiv \delta \pmod k$. Our goal now is to make $k|\alpha$ and $k|\gamma$, this would mean that k divides all exponents in the action of $S^1_{a,b}$, and therefore, has ineffective kernel $\mathbb{Z}_k$, which we can get rid of.

Let $r \equiv \gamma - \alpha \pmod k$. Take $A = \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} \in GL_2(\mathbb{Z})$.

Apply rule 5 from Proposition 5.3.2, and we get $E^7_d//S^1_{a,b} = E^7_d//S^1_{a',b'}$, where $a' = (\alpha + r, \beta + r, rd), b' = (\gamma, \delta, \varepsilon)$. Applying rule 3 we get $a'' = (\alpha + r(1-d), \beta + r(1-d), 0), b'' = (\gamma - rd, \delta - rd, \varepsilon)$.

Observe that $a''_1 - b''_1 = \alpha - \gamma + r \equiv 0 \pmod k$. Since this is just a reparametrization of the torus, it still has the same ineffective kernel, so $k|a''_1 d - b''_1(d-1)$, but $a''_1 d - b''_1(d-1) = (a''_1 - b''_1)d + b''_1 \equiv b''_1 \pmod k$. Therefore, $k|b''_1$, and so k must also divide a''_i, b''_i for all i. So, we divide a'', b'' by k, and get rid of the ineffective kernel. □

5.3.4 Orbifold Groups at $\mathcal{C}_\sigma$ and $\mathcal{L}_{ij}$

We assume from now on that the action of T^2 on $SU(3)$ is effective. The following lemma is essential to understanding the orbifold group at points in $\mathcal{L}_{ij}$ in terms of the orbifold groups on the $\mathcal{C}_\sigma$'s it connects.

Lemma 5.3.5. *Let $\mathcal{L}_{ij}$ connect $\mathcal{C}_\sigma$ and $\mathcal{C}_\tau$, then (z,w) acts trivially on $U(2)_{ij}$ iff (z,w) acts trivially on T^2_σ and T^2_τ.*

Proof. One direction is trivial. If (z,w) acts trivially on $U(2)_{ij}$, then it acts trivially on every subset, in particular the two torii.

By conjugation, we may assume without loss of generality that $\sigma = Id, \tau = (12)$, so $\mathcal{L}_{ij} = \mathcal{L}_{33}$, and $U(2)_{ij}$ is the standard embedding of $U(2)$ into $SU(3)$. Now assume that (z,w) acts trivially on T^2_{Id} and $T^2_{(12)}$. Since $I \in T^2_{Id}$, and $(z,w) \star X = \text{diag}(u_1,u_2,u_3) \cdot X \cdot \text{diag}(v_1,v_2,v_3)^{-1}$, if follows that $v_i = u_i$. Now observe that $A_{(12)} = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \in T^2_{(12)}$. Hence $(z,w) \star A_{(12)} = A_{(12)}$, implies that $u_1\overline{u_2} = 1$, so $u_1 = u_2$. Therefore, the action of (z,w) becomes $(z,w) \star X = \text{diag}(u_1,u_1,u_3) X \text{diag}(u_1,u_1,u_3)^{-1}$. Thus, (z,w) fixes all of $U(2)_{33}$ as well. □

The following results all assume that the action of S^1 or T^2 is effective.

Theorem 5.3.6. *Let $\Gamma^{p,q}_\sigma$ denote the orbifold group of $E^7_{p,q}$ along C_σ. Then $\Gamma^{p,q}_\sigma$ is a cyclic group of order* $\gcd(p_1 - q_{\sigma(1)}, p_2 - q_{\sigma(2)})$.

Proof. Let $z \in S^1$ be an element that fixes T^2_σ, then $z^{p_i - q_{\sigma(i)}} = 1$ for $i = 1,2,3$. In particular, if $r = \gcd(p_1 - q_{\sigma(1)}, p_2 - q_{\sigma(2)})$, then $z^r = 1$, and in fact, any z satisfying $z^r = 1$ fixes T^2_σ. Therefore, $\Gamma^{p,q}_\sigma = \mathbb{Z}_r$. □

Theorem 5.3.7. *For $\mathcal{O}^{a,b}_{p,q}$, the orbifold group at $\mathcal{C}_\sigma$ denoted by Γ_σ has order*

$$N_\sigma = \left|(p_1 - q_{\sigma(1)})(a_2 - b_{\sigma(2)}) - (a_1 - b_{\sigma(1)})(p_2 - q_{\sigma(2)})\right|.$$

Let $r_\sigma = \gcd(|\Gamma^{a,b}_\sigma|, |\Gamma^{p,q}_\sigma|)$, then $\Gamma_\sigma = \mathbb{Z}_{r_\sigma} \oplus \mathbb{Z}_{N_\sigma/r_\sigma}$. In particular, Γ_σ is non-cyclic iff the orders of $\Gamma^{a,b}_\sigma$ and $\Gamma^{p,q}_\sigma$ are not relatively prime.

Remark 5.3.1. In particular, it follows that if $\gcd(|\Gamma^{a,b}_\sigma|, |\Gamma^{p,q}_\sigma|) > 1$, then Γ_σ is non-cyclic, and so there is no Eschenburg 7-manifold $E^7_{u,v}$ such that $\mathcal{O}^{a,b}_{p,q} = E^7_{u,v}//S^1$.

Proof. The order of Γ_σ follows immediately from Proposition 3.7 in [FZ07]. All that remains to show is the assertion about its group structure.

Since $\Gamma_\sigma \subset T^2$, we conclude that it is either cyclic or a direct sum of two cyclic groups. It is clearly non-cyclic iff it has a subgroup of the form $\mathbb{Z}_n \oplus \mathbb{Z}_n$. We will show that this occurs iff n divides both $|\Gamma^{a,b}_\sigma|$ and $|\Gamma^{p,q}_\sigma|$.

If n divides both $|\Gamma^{a,b}_\sigma|$ and $|\Gamma^{p,q}_\sigma|$, then clearly any element of the form $(z, w) = (e^{2k\pi i/n}, e^{2l\pi i/n})$ fixes T^2_σ, and so $\mathbb{Z}_n \oplus \mathbb{Z}_n \subset \Gamma_\sigma$. Conversely, suppose that $\mathbb{Z}_n \oplus \mathbb{Z}_n \subset \Gamma_\sigma$, then we must have at least n^2 elements $(z, w) \in T^2$ that satisfy $(z^n, w^n) = (1, 1)$. However, there are precisely n^2 such elements, which implies that all of them act trivially on T^2_σ. In particular, $(e^{2\pi i/n}, 1)$ and $(1, e^{2\pi i/n})$ fix T^2_σ, so $e^{2\pi i/n}$ is both in $\Gamma^{a,b}_\sigma$ and in $\Gamma^{p,q}_\sigma$. Therefore, n divides the order of both groups. This completes the proof. □

Corollary 5.3.8. *Let $S^1_{a,b}$ act on E^7_d, then the orders of the orbifold groups are given by*

σ	N_σ
id	$\lvert(\alpha-\beta)-(\gamma-\delta)\rvert$
(12)	$\lvert(\alpha-\beta)+(\gamma-\delta)\rvert$
(13)	$\lvert\gamma+d(\beta-\delta)\rvert$
(123)	$\lvert\gamma+d(\alpha-\delta)\rvert$
(132)	$\lvert\delta+d(\beta-\gamma)\rvert$
(23)	$\lvert\delta+d(\alpha-\gamma)\rvert$

Table 5.1: $\mathcal{C}_\sigma$ singularities in $E^7_d/S^1_{a,b}$

Where $a=(\alpha,\beta,0)$ *and* $b=(\gamma,\delta,\alpha+\beta-\gamma-\delta)$.

This corollary follows from Theorem 5.3.7. Alternatively, the same results can be obtained from Proposition 3.7 in [FZ07].

Theorem 5.3.9. *Let* $S^1_{a,b}$ *act on* E^7_d *with* $a=(\alpha,\beta,0)$ *and* $b=(\gamma,\delta,\alpha+\beta-\gamma-\delta)$. *Then, the orbifold groups* Γ_{ij} *of* $\mathcal{L}_{ij}$ *is cyclic with order* N_{ij} *given by the following table:*

i,j	N_{ij}
1,1	$((\alpha-\beta)-(\gamma-\delta),\delta+d(\alpha-\gamma))$
1,2	$((\alpha-\beta)+(\gamma-\delta),\gamma+d(\alpha-\delta))$
1,3	$(\delta-\gamma,\delta+d(\beta-\gamma))$
2,1	$((\alpha-\beta)+(\gamma-\delta),\delta+d(\beta-\gamma))$
2,2	$((\alpha-\beta)-(\gamma-\delta),\gamma+d(\beta-\delta))$
2,3	$(\delta-\gamma,\delta+d(\alpha-\gamma))$
3,1	$(\alpha-\beta,\gamma+d(\alpha-\delta))$
3,2	$(\alpha-\beta,\delta+d(\alpha-\gamma))$
3,3	$(\alpha-\beta,\gamma-\delta)$

Table 5.2: $\mathcal{L}_{ij}$ singularities in $E^7_d/S^1_{a,b}$

Proof. This is an immediate consequence of part (c) of Proposition 3.7 of [FZ07]. □

5.3.5 Corrections to Theorem C [FZ07]

In this section we examine Theorem C of [FZ07] and provide both corrections and improvements to it. We restate Theorem D here for completeness.

Theorem 5.3.10. *Let E_d be a cohomogeneity one Eschenburg manifold, $d \geq 3$, equipped with a positively curved Eschenburg metric. Then:*

i) If S^1 acts on E^7_d by isometries, then there are at minimum 3 singular points, in particular, if exactly two C_σ's are singular, then the $\mathcal{L}_{ij}$ connecting them is *also singular.*

In the following particular examples the singular locus of the isometric circle action $S^1_{a,b}$ on E_d consists of:

ii) *A smooth totally geodesic 2-sphere with orbifold group $\mathbb{Z}_{d+1}$ if $a = (0, -1, 1)$ and $b = (0, 0, 0)$;*

iii) *When $a = (0, 1, 1)$and $b = (2, 0, 0)$, the singular locus consists of four point with orbifold groups $\mathbb{Z}_3, \mathbb{Z}_{d+1}, \mathbb{Z}_{d+1}, \mathbb{Z}_{2d+1}$, and the following orbifold groups on spheres:*

If $3|(d+1)$, then the first 2 points are connected by a totally geodesic 2-sphere with orbifold group $\mathbb{Z}_3$.

If $3|(d-1)$, then the first and the fourth points are connected by a totally geodesic 2-sphere with orbifold group $\mathbb{Z}_3$.

If $2|(d+1)$, then the second and the third points are connected by a totally geodesic 2-sphere with orbifold group $\mathbb{Z}_2$.

iv) *A smooth totally geodesic 2-sphere with orbifold group $\mathbb{Z}_{d-1}$ if $a = (0, 1, 1)$ and $b = (0, 0, 2)$.*

v) *Three isolated singular points with orbifold groups $\mathbb{Z}_{2d-3}$, $\mathbb{Z}_{d^2-d-1}$, and $\mathbb{Z}_{d^2-d-1}$ if $a = (0, d-1, 0)$ and $b = (1, d-1, -1)$.*

Proof. Parts 2-5 are direct application of Theorem 5.3.9 and Corollary 5.3.8. Part 1 deserves a special mention:

Assume that $S^1_{a,b}$ acts on E^7_d in such a way that at most two of N_σ's are not 1 (i.e. at most two $\mathcal{C}_\sigma$'s are singular). We start with a lemma:

Lemma 5.3.11. *If $\alpha = \beta$ or $\gamma = \delta$, then the singular locus of $E^7_d//S^1_{a,b}$ consists of smooth totally geodesic 2-spheres. Where $a = (\alpha, \beta, 0)$ and $b = (\gamma, \delta, \alpha + \beta - \gamma - \delta)$.*

Proof. Suppose $\alpha = \beta$, and (z, w) fixes T^2_σ, then look at $U(2)_{33}T^2_\sigma$. $\alpha = \beta$ implies that the matrix acting on the left is of the form $\mathrm{diag}(u, u, v)$, which commutes with $U(2)_{33}$. Therefore, (z, w) fixes $U(2)_{33}T^2_\sigma$.

If $\gamma = \delta$, apply the same argument to $T^2_\sigma U(2)_{33}$.

If $\mathcal{L}_{ij}$ is not of the form above and is singular, then so is one of the subspaces $U(2)_{33}U(2)_{ij}$ and $U(2)_{ij}U(2)_{33}$, which contradicts the orbifold structure of the singular locus. □

We now split the N_σ's into 3 pairs using Corollary 5.3.8:

- If $N_{id} = N_{(12)} = 1$, then either $\alpha = \beta$ or $\gamma = \delta$;
- If $N_{(13)} = N_{(123)} = 1$, then $\alpha = \beta$ (since $d \geq 3$);
- If $N_{(23)} = N_{(132)} = 1$, then $\alpha = \beta$ (since $d \geq 3$).

Since at most two N_σ's are not 1, we see that at least one of the above cases must occur. Therefore, by the lemma above, we see that $E^7_d//S^1_{a,b}$ has singular locus consisting of smooth totally geodesic 2-spheres.

It remains to show that there is no free action of $S^1_{a,b}$ on E^7_d.

Suppose that the action is free, then $N_{(13)} = N_{(123)} = 1$, so $\alpha = \beta$. Furthermore, $N_{(13)} = N_{(132)} = 1$, so we get $(d+1)(\gamma - \delta)$ is either -2, 0 or 2, but $d \geq 3$, so we must have $\gamma = \delta$, which implies $N_{id} = 0$, so the quotient $E^7_d//S^1_{a,b}$ is not an orbifold. □

As a corollary, we get

Remark 5.3.2. Parts 1 and 4 of Theorem C from [FZ07] hold, but parts 2 and 3 are false.

5.3.6 Curvature

The goal of this section is to prove Theorem E, which we restate here:

Theorem. *Given an orbifold $\mathcal{O}^{a,b}_{p,q}$ which has positive curvature induced by a Cheeger deformation along $U(2)$, there exists $E^7_{u,v}$ (either a manifold or an orbifold) such that $\mathcal{O}^{a,b}_{p,q} = E^7_{u,v}//S^1$ and $E^7_{u,v}$ has positive curvature induced by Cheeger deformation along the same $U(2)$.*

Equivalently, there exist $\lambda, \mu \in \mathbb{Z}$ relatively prime such that $E^7_{\lambda p+\mu a, \lambda q+\mu b}$ is positively curved.

The following is an example of why this theorem is non-trivial.

Example 5.3.1. Consider the Eschenburg orbifold $\mathcal{O}^{a,b}_{p,q}$ given by $a = (-2, 0, 2), b = (-3, 1, 2), p = (-4, 0, 2)$ and $q = (-5, 3, 0)$. From the work of Eschenburg, it follows that deforming by $U(2)$ does not result in a metric of positive curvature on either

$E^7_{p,q}$ or $E^7_{a,b}$. However, we can re-write this orbifold as $\mathcal{O}^{p',q'}_{a',b'}$, where $a' = a, b' = b$ and $p' = 2a - p = (0, 0, 2), q' = 2b - q = (-1, -1, 4)$, this is the same orbifold according to Proposition 5.3.2. Additionally now $E^7_{p',q'} = E^7_2$ admits positive curvature.

Before proving this theorem, we state a lemma that limits the parametrizations we need to consider:

Lemma 5.3.12. *Let $\mathcal{O}^{a,b}_{p,q}$ be a 6-dimensional Eschenburg space, then there exist a reparametrization p', q', a', b' (not necessarily effective) satisfying one of the following:*

- $p'_1 = p'_2 = p'_3 = 0$, *or*
- $p'_1 = p'_3 = a'_1 = 0, p'_2 = a'_2 = a'_3 = n$ *for some* $n \in \mathbb{Z}^+$.

Proof. We begin with an intermediate reparametrization p^0, q^0, a^0, b^0 where $p^0_1 = a^0_1 = 0$. Now consider $\Delta = a^0_2 p^0_3 - a^0_3 p^0_2$.

Suppose that $\Delta = 0$, then we want to show that we have the first scenario. Additionally suppose $p^0_2 \neq 0$ (if $p^0_2 = 0, p^0_3 \neq 0$ just apply $(12) \in S_3$ to both p and a to get $p^0_2 = 0$, and if both are 0, we already have case 1).

If $a^0_3 \neq 0$, then $a^0_2 \neq 0, p^0_3 \neq 0$. This implies that $p^0_2 = ta^0_2, p^0_3 = ta^0_3$ for some $t \in \mathbb{Q} \setminus \{0\}$, so there exist $m, n \in \mathbb{Z}$ relatively prime so that $mp^0_i + na^0_i = 0$ for all i ($t = m/n$). Now consider $m', n' \in \mathbb{Z}$ such that $m'm - n'n = 1$, define $p' = mp^0 + na^0, a' = n'p^0 + m'a^0$ (analogously for q', b'), then $p'_i = 0$ for all i and we have case 1.

If $a_3^0 = 0$, then $a_2^0 = 0$ or $p_3^0 = 0, a_2^0 \neq 0$, if the former, then let $p' = a^0, a' = p^0$ giving us case 1. In the latter case, let m, n be such that $mp_2^0 + na_2^0 = 0$. Now consider $m', n' \in \mathbb{Z}$ such that $m'm - n'n = 1$, define $p' = mp^0 + na^0, a' = n'p^0 + m'a^0$ (analogously for q', b'), then $p'_i = 0$ for all i and we have case 1.

Next suppose that $\Delta \neq 0$, our goal is to show that we have the second scenario. Consider $\widetilde{p} = -a_3^0 p + p_3^0 a, \widetilde{a} = -a_2^0 p + p_2^0 a$ (similarly for q, b). Then $\widetilde{p}_1 = \widetilde{p}_3 = \widetilde{a}_1 = \widetilde{a}_2 = 0$, let $n = lcm(\widetilde{p}_2, \widetilde{a}_3)$ and $k = n/\widetilde{p}_2, l = n/\widetilde{a}_3$. Then define $p' = k\widetilde{p}, a' = k\widetilde{p} + l\widetilde{a}$. This gives us $p' = (0, n, 0), a' = (0, n, n)$. □

We also prove the explicit conditions in terms of a, b, p, q for when the orbifold $\mathcal{O}_{p,q}^{a,b}$ has positive sectional curvature:

Proposition 5.3.13. *Let $\mathcal{O}_{p,q}^{a,b}$ be as above, and the metric being one given by Cheeger deformation along $U(2)$. Then, $\mathcal{O}_{p,q}^{a,b}$ is positively curved iff for each $t \in [0,1]$ and each triple (η_1, η_2, η_3) satisfying $\eta_i \geq 0$ and $\sum \eta_i = 1$ we have both*

$$(1-t)b_1 + tb_2 \neq \sum \eta_i a_i \qquad OR \qquad (1-t)q_1 + tq_2 \neq \sum \eta_i p_i \tag{5.1}$$

and

$$b_3 \neq \sum \eta_i a_i \qquad OR \qquad q_3 \neq \sum \eta_i p_i \tag{5.2}$$

Remark 5.3.3. We point out that which half of each condition is satisfied can in general depend on the choice of $\eta_i' s$ and t.

Proof. This is a fairly straightforward application of Eschenburg's original results on the curvature of Eschenburg spaces. In particular, we know that $\sec \sigma = 0$ iff one of the following vectors is in σ

$$Y_3 = \begin{pmatrix} i & & \\ & i & \\ & & -2i \end{pmatrix} \qquad \mathrm{Ad}(k)Y_1 = k \begin{pmatrix} -2i & & \\ & i & \\ & & i \end{pmatrix} k^{-1} \qquad (k \in U(2)).$$

Condition 2 corresponds to verifying that Y_3 is not horizontal, and condition 1 corresponds to verifying that $\mathrm{Ad}(k)Y_1$ is not horizontal.

Let $V_{a,b}(X), V_{p,q}(X)$ be the vectors tangent to the action of $S^1_{a,b}$ and $S^1_{p,q}$ respectively at the point $X \in SU(3)$. It is easy to see that $V_{a,b}(X) = X^{-1} \cdot A \cdot X - B$, where $A = \mathrm{diag}(a_1 i, a_2 i, a_3 i)$ and $B = \mathrm{diag}(b_1 i, b_2 i, b_3 i)$ (similarly for $V_{p,q}(X)$). Let

$$X = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}$$

To verify that Y_3 is not horizontal, we need only consider the diagonal entries of $V_{a,b}(X), V_{p,q}(X)$, which are of the form

$$\left(\sum_{j=1}^{3} |x_{j1}|^2 a_j - b_1\right) i, \qquad \left(\sum_{j=1}^{3} |x_{j2}|^2 a_j - b_2\right) i, \qquad \left(\sum_{j=1}^{3} |x_{j3}|^2 a_j - b_3\right) i$$

Y_3 is orthogonal to $V_{a,b}(X)$ iff $b_3 = \sum |x_{j3}|^2 a_j$. Similar condition holds for Y_3 being orthogonal to $V_{p,q}(X)$. Therefore, Y_3 is not horizontal iff condition (2) holds.

We will approach the question of whether $\mathrm{Ad}(k)Y_1$ is horizontal differently. First observe that $\langle \mathrm{Ad}(X)^{-1}A - B, \mathrm{Ad}(k)Y_1\rangle = \langle \mathrm{Ad}(Xk)^{-1}A - \mathrm{Ad}(k)^{-1}B, Y_1\rangle$. Let

$$Xk = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}, \qquad k = \begin{pmatrix} c\alpha & c\beta & \\ -c\overline{\beta} & c\overline{\alpha} & \\ & & \overline{c}^2 \end{pmatrix}.$$

Then, the diagonal entries of $\mathrm{Ad}(Xk)^{-1}A - \mathrm{Ad}(k)^{-1}B$ are

$$\left[\sum_{j=1}^{3} |x_{j1}|^2 a_j - \big(b_1|\alpha|^2 + b_2|\beta|^2\big)\right] i, \qquad \left[\sum_{j=1}^{3} |x_{j2}|^2 a_j - \big(b_1|\beta|^2 + b_2|\alpha|^2\big)\right] i,$$
$$\left[\sum_{j=1}^{3} |x_{j3}|^2 a_j - b_3\right] i$$

Taking the inner product with Y_1, we get:

$$3\left[|\alpha|^2 b_1 + |\beta|^2 b_2 - \sum_{j=1}^{3} |x_{j1}|^2 a_j\right] + \sum_{j=1}^{3} a_j - \sum_{j=1}^{3} b_j.$$

Letting $\eta_j = |x_{j1}|^2, t = |\beta|^2$, we get that $\mathrm{Ad}(k)Y$ is orthogonal to $V_{a,b}^X$ iff

$$(1-t)b_1 + tb_2 = \sum_{j=1}^{3} \eta_j a_j$$

Obtaining a similar formula for $V_{p,q}^X$, we conclude that $\mathrm{Ad}(k)Y$ is not horizontal iff condition (1) holds. □

With these results established, we prove the main result:

Proof of Theorem E. This proof is organized according to the cases given in Lemma 5.3.12.

Case 1: $p_1, p_2, p_3 = 0$.

First suppose that q_1, q_2 are both positive or both negative, then $E_{p,q}^7$ has positive curvature. Therefore, we will assume, without loss of generality, that $q_1 \le 0 \le q_2$.

Next suppose that $b_2 - b_1 = q_2 - q_1$. If $b_1 - q_1 \in [\min a_i, \max a_i]$, then consider $\alpha \in [0,1], \eta_i \geq 0, \sum \eta_i = 1$ such that $b_1 - q_1 = \sum \eta_i a_i$, and $\alpha q_1 + (1-\alpha) q_2 = 0 = \sum \eta_i p_i$, then $\alpha b_1 + (1-\alpha) b_2 = \alpha q_1 + (1-\alpha) q_2 + \sum \eta_i a_i = \sum \eta_i a_i$, which violates the positivity of sectional curvature for $\mathcal{O}^{a,b}_{p,q}$. Now suppose that $b_1 - q_1 \notin [\min a_i, \max a_i]$, then there exists $n \in \mathbb{Z}^+$ satisfying $q_1 + t(b_1 - q_1), q_2 + t(b_2 - q_2)$ both $< \min t a_i$ or both $> \max t a_i$. Consider $u = (1-t)p + ta$ and $v = (1-t)q + tb$, then $E^7_{u,v}$ has positive sectional curvature.

Now suppose that $b_2 - b_1 \neq q_2 - q_1$. Then there exist $m, n \in \mathbb{Z}$ such that $nq_1 + m(b_1 - q_1) = nq_2 + m(b_2 - q_2)$. If $nq_1 + m(b_1 - q_1) \notin [\min m a_i, \max m a_i]$, then take $u = (n-m)p + ma$ and $v = (n-m)q + mb$ to get $E^7_{u,v}$ with positive curvature. Otherwise we have $nq_1 + m(b_1 - q_1) \in [\min m a_i, \max m a_i]$, then let η_i be such that $nq_1 + m(b_1 - q_1) = m \sum \eta_i a_i$ and α such that $\alpha q_1 + (1-\alpha) q_2 = 0 = \sum \eta_i p_i$. This implies that $m\left(\alpha b_1 + (1-\alpha) b_2\right) = m \sum \eta_i a_i$. So either $\alpha b_1 + (1-\alpha) b_2 = \sum \eta_i a_i$, in which case we don't have sec > 0 on $\mathcal{O}^{a,b}_{p,q}$ or $m = 0$ and so $q_1 = q_2 = 0$, however, this implies $q_3 = 0$ as well, and so we have a degeneracy.

Case 2: $p_1 = p_3 = a_1 = 0, p_2 = a_2 = a_3 = n > 0$.

Subcase 2a: $b_1 - q_1 = b_2 - q_2 = k$.

If $k < 0$ or $k > n$, there exists $t \in \mathbb{Z}^+$ such that $q_1 + t(b_1 - q_1)$ an $q_2 + t(b_2 - q_2)$ are both < 0 or $> tn$ respectively. Then take $u = (1-t)p + ta, v = (1-t)q + tb$ and we get that $E^7_{u,v}$ has sec > 0.

Next, without loss of generality, assume $q_1 \leq q_2$. If $[q_1, q_2] \cap [0, n] = \emptyset$, then $E^7_{p,q}$

has positive curvature, otherwise let $m = \min([q_1, q_2] \cap [0, n]) = \max\{q_1, 0\}$. Suppose that $b_1 - q_1 \in [0, n-m] \subset [0, n]$, then pick η_i such that $\eta_2 = m/n, n\eta_3 = b_1 - q_1$ and pick α such that $\alpha q_1 + (1-\alpha) q_2 = m$. This implies that $\sum \eta_i p_i = m = \alpha q_1 + (1-\alpha) q_2$ and $\sum \eta_i a_i = m + (b_1 - q_1) = \alpha b_1 + (1-\alpha) b_2$, so $\mathcal{O}^{a,b}_{p,q}$ does not have $\sec > 0$. Next observe that if $q_1 \leq 0 \leq q_2$, then $m = 0$ and so every possible value of $b_1 - q_1$ has been handled. Finally, suppose that $q_1 \in (0, n]$, then $m = q_1$, so we need only consider $b_1 - q_1 \in (n-m, n]$, but this implies that $n < b_1 \leq b_2$, so $E^7_{a,b}$ has positive curvature.

Subcase 2b: Without loss of generality, $b_1 - q_1 < b_2 - q_2$. This implies that there exist $k, l \in \mathbb{Z}$ such that $kq_1 + l(b_1 - q_1) = kq_2 + l(b_2 - q_2)$, $k > 0$. Additionally, let $t_0 = kq_1 + l(b_1 - q_1)$.

Suppose that $t_0 \notin [\min\{kp_i + l(a_i - p_i)\}, \max\{kp_i + l(a_i - p_i)\}]$. Then, let $u = (k-l)p + la, v = (k-l)q + lb$ to get $\sec > 0$ on $E^7_{u,v}$.

Lemma 5.3.14. *Consider all the possible values η_i such that $kq_1 + l(b_1 - q_1) = \sum \eta_i [kp_i + l(a_i - p_i)]$, and let η_3^m, η_3^M denote the smallest and largest values of η_3 respectively.*

If $[b_1 - q_1, b_2 - q_2] \cap [n\eta_3^m, n\eta_3^M] \neq \emptyset$, then $\mathcal{O}^{a,b}_{p,q}$ does not have $\sec > 0$.

Proof. Pick α such $\alpha(b_1 - q_1) + (1-\alpha)(b_2 - q_2) = n\eta_3'$, then $\alpha k q_1 + (1-\alpha) k q_2 = \sum k\eta_i' p_i$, so $\alpha q_1 + (1-\alpha) q_2 = \sum \eta_i' p_i$. We also get $\alpha b_1 + (1-\alpha) b_2 = \sum \eta_i' a_i$, so we do not have $\sec > 0$. ☐

The table below demonstrates the possible relations between k, l, t_0 and the corresponding η_3^m, η_3^M:

Relation	$n\eta_3^m$	$n\eta_3^M$
$0 < k < t_0/n \leq l$	$\frac{t_0-kn}{l-k}$	$\frac{t_0}{l}$
$0 \leq t_0/n \leq k,\, 0 < l$	0	$\frac{t_0}{l}$
$0 \leq t_0/n \leq k,\, l \leq 0$	0	$\frac{t_0-kn}{l-k}$
$l \leq t_0/n \leq 0 < k,\, l < 0$	$\frac{t_0}{l}$	$\frac{t_0-kn}{l-k}$

Table 5.3: Bounds on η_3

If $b_i - q_i > t_0/l$ and $l \geq 0$, then $q_i < 0 = \min p_i$, and so $E^7_{p,q}$ has sec > 0.

If $b_i - q_i < t_0/l$ and $l < 0$, then $q_i < 0 = \min p_i$, and so $E^7_{p,q}$ has sec > 0.

If $b_i - q_i < 0$, then if we take $N > 0$ sufficiently large, we get $q_i + N(b_i - q_i) < 0$ for $i = 1, 2$, and so $E^7_{u,v}$ with $u = (1 - N)p + Na, v = (1 - N)q + Nb$ has sec > 0.

If $b_i - q_i < (t_0 - kn)/(l - k)$ and $l > k > 0$, then $b_i > n = \max a_i$, and so $E^7_{a,b}$ has sec > 0.

If $b_i - q_i > (t_0 - kn)/(l - k)$ and $l \leq 0 < k$, then $b_i > n = \max a_i$, and so $E^7_{a,b}$ has sec > 0. □

Remark 5.3.4. We also note that if $\mathcal{O}^{a,b}_{p,q}$ has positive sectional curvature, then there exist a', b', p', q' such that $\mathcal{O}^{a',b'}_{p',q'} = \mathcal{O}^{a,b}_{p,q}$ and $E^7_{a',b'}, E^7_{p',q'}$ both have positive sectional curvature. This is achieved by finding p', q' in accordance with the theorem, and taking a circle $S^1_{a',b'} \subset T^2$ sufficiently close to $S^1_{p',q'}$.

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